EXTENSIONS OF COMPLETELY 0-SIMPLE SEMIGROUPS
BY COMPLETELY 0-SIMPLE SEMIGROUPS

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Let S and T be disjoint semigroups, T having a zero element, O'. A semigroup (V, O) is called an extension of S by T if it contains S as an ideal and if the Rees factor semigroup V/S is isomorphic to T. We shall say that V is determined by a partial homomorphism if there exists a partial homomorphism \( A \rightarrow \overline{A} \) of \( T\setminus O' \) into S such that \( A \circ B = AB \) if \( AB \neq O' \), \( A \circ B = AB \) if \( AB = O' \), \( A \circ s = \overline{As} \), \( s \circ A = s\overline{A} \), and \( s \circ t = st \) where \( s, t \) in S and the operations in S and T are denoted by juxtaposition. The purpose of this note is to give a necessary and sufficient condition that V be determined by a partial homomorphism when S is a completely 0-simple semigroup and T is a completely 0-simple semigroup. Since these partial homomorphisms are known mod group homomorphisms [1, p. 109, Theorem 3.14], our extensions may be given an explicit form. A corollary to our theorem will include an important theorem due to W. D. Munn [1, p. 143, Theorem 4.22]. Our result should have important applications to the study of finite semigroups and to semigroups with some finiteness condition.

If S is any subset of a semigroup, \( \xi(S) \) will denote the set of idempotents of S and \( S^* \) will denote the set of nonzero elements of S. \( \mathfrak{R}, \mathfrak{L}, \mathfrak{D} \) and \( \mathfrak{E} \) will denote Green's relations [1, p. 47]. If \( a \in S \), \( R_a \) will denote the \( \mathfrak{R} \)-class containing a. If \( e \) and \( f \) are idempotents and \( ef = fe = e \), we say \( e \) is under \( f \) and write \( e \prec f \). Basic definitions are given in [1]. Likewise references to the fundamental work of Clifford, Green, and Munn will be found in [1].

**Lemma.** Let V be an extension of a completely 0-simple semigroup S by a completely 0-simple semigroup T, and let 0 be the zero of S (hence, 0 is also the zero of V). If there is some \( E \in \xi(T^*) \) such that \( ESE = 0 \), then V is given by the partial homomorphism \( \xi: T^* \rightarrow S \) which maps every element of \( T^* \) to 0.

**Proof.** Let \( E \in \xi(T^*) \) such that \( ESE = 0 \). If \( F \in \xi(T^*) \), there exists \( Y \in T^* \) such that \( E \mathfrak{R} Y \) and \( Y \mathfrak{L} F \) [1, p. 79, Theorem 2.51]. Thus,
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Let \( EY = Y \) and there exists \( U \in T^* \) such that \( UY = F \). Let \( s \in S \). Now, \( ESY = ESEY = 0 \), \( EYSY = 0 \), \( USY = 0 \), \( USY = 0 \), \( FSF = 0 \), and \( FSF = 0 \). Thus, \( FSF = 0 \) for all \( F \in \varepsilon(T^*) \). Now suppose that \( A \in R_E \cap L_F \) where \( E, F \in \varepsilon(T^*) \). Let \( h \in S \) and \( hA \in Lg \) for some \( g \in E_S \). Since \( uhA = g \) for some \( u \in S \), \( gF = uhAF = g \). Thus, \( Fg = FgF = 0 \). Hence \( gF = g0 = 0 = gg = g \) and \( hA = 0 \). Similarly \( Ah = 0 \). Now define \( A \xi = 0 \) for all \( A \in T^* \). Clearly \( \xi \) is a partial homomorphism of \( T^* \) into \( S \). If \( AB = s \in S \), \( AB = (EA)B = Es = 0 = (A\xi)(B\xi) \).

**Theorem.** An extension \( V \) of a completely 0-simple semigroup \( S \) by a completely 0-simple semigroup \( T \) is given by a partial homomorphism if and only if under each nonzero idempotent of \( T \) there exists at most one nonzero idempotent of \( S \).

**Proof.** Suppose that under each nonzero idempotent of \( T \) there exists at most one nonzero idempotent of \( S \). By virtue of the lemma, we may assume that \( ESE \neq 0 \) for all \( E \in \varepsilon(T^*) \). We will first show that under each nonzero idempotent \( E \in T^* \), there exists a unique idempotent \( e \in S^* \) and that \( ESE = H_e U_0 \). Let \( a \in ESE \) and \( a \neq 0 \). There exists \( x \in ESE \) such that \( axa = a \). Thus \( ax = e \in E(S^*) \) and \( e \leq E \), i.e., \( e \) is the unique idempotent of \( S^* \) which lies under \( E \). Hence \( ax = e \) and \( ESE \subseteq H_e U_0 \). If \( b \in H_e \), \( bE = beE = be = b = Eb \) and \( ESE = H_e U_0 \). Let \( A \in R_E \cap L_F \) where \( E, F \in \varepsilon(T^*) \), and let \( e \) and \( f \) be the unique idempotents of \( S^* \) under \( E \) and \( F \) respectively. We will show that if \( h \in S, hA \neq 0 \) implies that \( hA \in L_f \) and \( Ah \neq 0 \) implies that \( Ah \in R_e \). We consider only the first case, the other case being similar. Now, \( hA \in Lg \) for some \( g \in \varepsilon(S^*) \). As in the proof of the lemma, we show that \( gF = g \). There exists \( k \in S^* \) such that \( gk = k \) and \( k \neq f \). Thus, \( kF = kgF = kg = fk = FfF = Fk \) and \( k \in FSF \). Hence \( k \in H_f \) and \( hA \in L_f \). Next suppose that \( A \) is also an element of \( R_E \cap L_F \), where \( E' \) and \( F' \in \varepsilon(T^*) \) and let \( e' \) and \( f' \) denote the unique idempotents under \( E' \) and \( F' \) respectively. We will show that \( eAf = e'Af' \) and hence it will follow that we may write \( A^e = eAf \) where \( \xi \) is a single valued mapping of \( T^* \) into \( S \). We first note that \( Ff = F, F'F = F', fF' = f', \) and \( f'F = f' \). Thus, \( FSFF'FS' = FSFSF' \). Since \( f \in SFS, SFS = S, \) and \( f = FfF' \in FSF, F'F' = H_f \cup 0 \). Thus \( H_f H_f \neq 0 \) and \( f^f = H_f H_f \). In an analogous manner, we show that \( e \neq e' \). If \( e' E = 0, e' EE' = 0 \) and \( e' = 0 \). Hence \( e' E = 0 \) and similarly \( Ff' = 0 \). Thus, \( e' Af' = e' EA Ff' f = e' E (Ff')f = e' E Af \). We next show that \( \xi \) is a partial homomorphism of \( T^* \) into \( S \). Let \( B \in R_g \cap L_H \) where \( G, H \in \varepsilon(T^*) \), and let \( g \) and \( h \) denote the unique idempotents of \( S^* \) under
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$G$ and $H$ respectively. If $AB \neq O'$, $AB \in R_E \cap L_H$ [1, p. 79, Theorem 2.52]. Thus $(AB)\xi = eABh$ and $A\xi B\xi = eAfgBh = [(eA)f][g(Bh)]$. Since $A \in E$, $eA \in e$ and $eA \neq 0$. Similarly, $Bh \neq 0$ and hence $(AB)\xi = A\xi B\xi$. Now, let $b \in S$. If $eAfb \neq 0$, $eAfb = ((eA)f)b = (eA)b = e(AB) = Ab$. If $Ab \neq 0$, we may reverse the steps. Thus $Ab = (A\xi)b$ in all cases. Similarly $bA = b(A\xi)$. Now suppose that $AB = s \in S$. Then, $s = sH = sh = (AB)h$. If $s \neq 0$, $A(Bh) = A\xi Bh = A\xi gBh = A\xi B\xi$ and $AB = A\xi B\xi \neq 0$. If $A\xi B\xi \neq 0$, $A\xi B\xi = ((eA)f)(g(Bh)) = eA(g(Bh)) = ((eA)g)(Bh) = (e(Ag))(Bh) = (Ag)(Bh) = A(g(Bh)) = A(Bh) = s$. Thus, in all cases, $AB = A\xi B\xi$.

Conversely, suppose that the extension $V$ is given by a partial homomorphism $\xi$ of $T^*$ into $S$. First suppose that $A\xi = 0$ for some $A \in T^*$. Let $U = \{A \in T^* : A\xi = 0\}$ and let $U' = U \cup 0$. Clearly, $U'$ is a nonzero ideal of $T$ and hence $U' = T$, i.e., $A\xi = 0$ for all $A \in T^*$. In this case, if $E \in \mathcal{E}(T^*)$, $0$ is the only idempotent under $E$. If $A\xi \neq 0$ for all $A \in T^*$, $E\xi$ is the unique idempotent of $S^*$ under $E$.

**Corollary.** An extension $V$ of a completely simple semigroup $S$ by a completely 0-simple semigroup $T$ is given by a partial homomorphism if and only if under each nonzero idempotent of $T$ there exists at most one idempotent of $S$.

**Remark.** In the statement of the theorem, we may replace $T$ by a regular semigroup with zero in which every nonzero idempotent is primitive [1]. This follows since such a semigroup is an orthogonal sum of completely 0-simple semigroups [2].

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**References**


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