FINITE INTERPOLATION FOR ANALYTIC FUNCTIONS WITH FINITE DIRICHLET INTEGRALS

MITSURU NAKAI

The finite interpolation problem for AD-functions on planar (zero genus) Riemann surfaces was completely solved by Sario [2] and Rodin [1]. We shall extend their result to the case of Riemann surfaces with finite genus.

**Theorem.** Let $R$ be an open Riemann surface of finite genus. Given a finite number of distinct points $\xi_k$ ($k = 1, 2, \cdots, n$) in $R$, local parameters $z_k$ at $\xi_k$ with $z_k(\xi_k) = 0$ ($k = 1, 2, \cdots, n$) and complex numbers $\alpha_k$ ($\nu = 0, 1, \cdots, m; k = 1, 2, \cdots, n$). Then there exists a bounded analytic function $f$ with finite Dirichlet integral on $R$ such that

$$\frac{d^n f}{dz_k^n}(\xi_k) = \alpha_k \quad (\nu = 0, 1, \cdots, m; k = 1, 2, \cdots, n)$$

if and only if $R$ does not belong to the class $0_{AD}$.

**Proof.** The necessity of the condition $R \not\in 0_{AD}$ is evident. We have to show the solvability of (1) under the condition $R \not\in 0_{AD}$. Since $R$ has finite genus, $R \not\in 0_{AD}$ implies the existence of a nonconstant ABD-function $F(z)$ on $R$. Let $R^*$ be a closed Riemann surface which contains $R$ as a subsurface. Choose a point $\xi_0$ in $R - \{\xi_1, \xi_2, \cdots, \xi_n\}$ such that $F(\xi_0) \neq F(\xi_k)$ ($k = 1, 2, \cdots, n$). For each fixed $k$ ($k = 1, 2, \cdots, n$), by Riemann-Roch's theorem, there exists a meromorphic function $r_k(z)$ on $R^*$ such that $r_k(z)$ has a simple pole at $\xi_k$ and a pole of order $n_k$ at $\xi_0$ and regular on $R^* - \{\xi_0, \xi_k\}$. Let $m_k$ be the order of zero of the function $\prod_{k=1}^{n} (F(z) - F(\xi_j))^{m+1}$ at $\xi_k$ and let $s = \max \{m_kn_k; k = 1, 2, \cdots, n\}$. Put

$$H(z) = (F(z) - F(\xi_0))^s \prod_{j=1}^{n} (F(z) - F(\xi_j))^{m+1},$$

which belongs to the class $ABD(R)$. By construction, $(d^{m_k}H/dz_k^{m_k})(\xi_k) \neq 0$ and $(z_kr_k)(\xi_k) \neq 0$. Hence for each $\nu$ ($\nu = 0, 1, \cdots, m$), we may set

$$H_{\nu}(z) = \left[ (\nu)! \cdot \frac{d^{m_k}H}{dz_k^{m_k}}(\xi_k)((z_kr_k)(\xi_k))^{m_k-\nu} \right]^{-1} \cdot (r_k(z))^{m_k-\nu} \cdot H(z).$$

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Again from the construction it is easy to see that $H_{\nu k}$ belongs to $\text{ABD}(R)$ and for each $k$ ($k = 1, 2, \ldots, n$),

$$H_{\nu k}(\zeta_j) = \delta_{k j} \quad (j = 1, 2, \ldots, n)$$

and moreover for each fixed $\nu$ ($\nu = 1, 2, \ldots, m$),

$$\frac{d^\mu H_{\nu k}}{d\zeta_j^\mu} (\zeta_j) = 0 \quad (\mu = 0, 1, \ldots, \nu - 1; j = 1, 2, \ldots, n),$$

$$\frac{d^\rho H_{\nu k}}{d\zeta_j^\rho} (\zeta_j) = \delta_{k j} \quad (j = 1, 2, \ldots, n).$$

Define $m + 1$ functions $P_\nu(z) (\nu = 0, \ldots, m)$ on $R$ inductively by

$$P_\nu(z) = P_{\nu-1}(z) + \sum_{j=1}^{n} \left( \alpha_{\nu j} - \frac{d^2 P_{\nu-1}}{dz_j^2} (\zeta_j) \right) H_{\nu j}(z) \quad (\nu = 0, \ldots, m)$$

with $P_{-1} = 0$. Then $f(z) = P_m(z)$ belongs to $\text{ABD}(R)$ and satisfies (1).

**Corollary.** Let $R$ be an open Riemann surface of finite genus not belonging to the class $0_{\text{AD}}$ and $\mathcal{F} = \mathcal{F}((\zeta_k), (z_k), (\alpha_{\nu k}))$ be the class of all $\text{AD}$-functions $f$ on $R$ satisfying the interpolating condition (1). Then the class $\mathcal{F}$ is not empty and there exists a unique function $f_0$ in $\mathcal{F}$ such that

$$D(f) = D(f_0) + D(f - f_0)$$

for any $f$ in $\mathcal{F}$ and a fortiori $f_0$ is the unique solution with minimum norm of the interpolation problem given by (1):

$$D(f_0) = \min \{ D(f); f \in \mathcal{F} \}.$$

**Proof.** For each closed parametric disk $K_k$ with local parameter $z_k$ ($k = 1, 2, \ldots, n$) and for any relatively compact parametric disk $U$ with local parameter $z$ such that $z_k \in U$ ($k = 1, 2, \ldots, n$), by the local subharmonicity of $|f''|^2$ for $f \in \mathcal{F}$ and Cauchy's inequalities, we can find a constant $c_U$ such that

$$|f_1(z) - f_2(z)|^2 + \sum_{k=1}^{n} \sum_{\nu=0}^{m} \left| \frac{d^2 f_1}{dz_k^{\nu}} (z_k) - \frac{d^2 f_2}{dz_k^{\nu}} (z_k) \right|^2 \leq c_U D(f_1 - f_2)$$

for any $z \in U$, $z_k \in K_k$ ($k = 1, 2, \ldots, n$) and $f_1, f_2 \in \mathcal{F}$. Let $\{f_n\}$ be a sequence such that $\{f_n\} \subset \mathcal{F}$ and $\lim_n D(f_n) = d = \inf \{ D(f); f \in \mathcal{F} \}$. Since $(f_n + f_{n+p})/2 \in \mathcal{F}$ and

$$D(f_n - f_{n+p}) = 2(D(f_n) + D(f_{n+p})) - 4 D \left( \frac{f_n + f_{n+p}}{2} \right) \leq 2(D(f_n) + D(f_{n+p})) - 4d,$$
we conclude that $\lim_n D(f_n - f_{n+p}) = 0$ for any \( p \). This with (2) gives the existence of a function \( f_0 \) in \( \mathcal{F} \) such that $\lim_n D(f_n - f_0) = 0$ so that $D(f_0) = d$. For any $f \in \mathcal{F}$ and any complex number \( \lambda, f_0 + \lambda(f - f_0) \in \mathcal{F} \). Hence $D(f_0 + \lambda(f - f_0)) \geq D(f_0)$. Whence it follows that $D(f_0, f - f_0) = 0$. Therefore $D(f) = D(f_0 + (f - f_0)) = D(f_0) + D(f - f_0) = d + D(f - f_0)$. Thus $D(f) = \lambda$ if and only if $f = f_0$.

References
