ON MAJORIZATION, FACTORIZATION, AND RANGE INCLUSION OF OPERATORS ON HILBERT SPACE

R. G. DOUGLAS

The purpose of this note is to show that a close relationship exists between the notions of majorization, factorization, and range inclusion for operators on a Hilbert space. Although fragments of these results are to be found scattered throughout the literature (usually buried in proofs), it does not seem to have been noticed how nicely they fit together to yield our theorems. We will also make an attempt at extending our result to the case of unbounded operators in the hope that it might be useful in establishing existence theorems for linear partial differential equations.

The author wishes to acknowledge that he discovered these relations in the study of an unpublished manuscript of deBranges and Rovnyak. Also, we acknowledge our indebtedness to P. Halmos for several conversations on this subject and note, in particular, that it was he who first noticed the equivalence of (1) and (3) in Theorem 1.

The Hilbert space considered can be either real or complex. We use [1] as our basic reference and use the definitions and notation therein with the following exception. For an operator A on the Hilbert space $\mathfrak{H}$ we will denote the range and null space of A by range $[A]$ and null $[A]$, respectively.

THEOREM 1. *Let A and B be (bounded) operators on the Hilbert space $\mathfrak{H}$. The following statements are equivalent:*

(1) range $[A] \subset$ range $[B]$;

(2) $AA^* \leqq \lambda^2 BB^*$ *for some* $\lambda \geqq 0$; *and*

(3) *there exists a bounded operator C on $\mathfrak{H}$ so that $A = BC$.*

Moreover, if (1), (2) *and* (3) *are valid, then there exists a unique operator C so that*

(a) $\|C\|^2 = \inf\{\mu \mid AA^* \leqq \mu BB^*\}$;

(b) null $[A]$ = null $[C]$; *and*

(c) range $[C] \subset$ range $[B^*]^-$.

PROOF. If $A = BC$, then

$$AA^* = BCC^*B^* = \|C\|^2 BB^* - B(\|C\|^2 I - CC^*)B^* \leqq \|C\|^2 BB^*,$$

so that (3) implies (2). Further, it is clear that (3) implies (1).

If we suppose that range $[A] \subset$ range $[B]$, then we can define an operator C_1 on $\mathfrak{H}$ as follows: for $f \in \mathfrak{H}$, we have $Af \in$ range $[A]$

Received by the editors June 28, 1965.

$\subset$range $[B]$, so that there exists $h \in$null $[B]^{\perp}$ for which $Bh = Af$. Set $C_1 f = h$. Then $A = BC_1$ and it remains only to prove that C_1 is bounded. Since C_1 is defined on all of $\mathfrak{H}$, to do this it suffices to show that C_1 has a closed graph.

If $\{[f_n, h_n]\}_{n=1}^{\infty}$ is a sequence of elements each in the graph of C_1 so that $\lim_{n\to\infty} [f_n, h_n] = [f, h]$, then $\lim_{n\to\infty} Af_n = Af$ and $\lim_{n\to\infty} Bh_n = Bh$. Thus, $Af = Bh$ and further, because null $[B]$ is closed, it follows that $h \in$null $[B]^{\perp}$ so that $C_1 f = h$. Hence C_1 has been shown to be bounded and (1) implies (3).

Lastly, suppose $AA^* \leqq \lambda^2 BB^*$ for some $\lambda \geqq 0$. Define a mapping D from range $[B^*]$ to range $[A^*]$ so that $D(B^*f) = A^*f$. Then D is well defined since

$$\|D(B^*f)\|^2 = \|A^*f\|^2 = (AA^*f, f) \leqq \lambda^2(BB^*f, f) = \lambda^2\|B^*f\|^2.$$

Hence, D can be uniquely extended to range $[B^*]^-$, and if we define D on range $[B^*]^{\perp}$ to be 0, then $DB^* = A^*$. If we set $C_2 = D^*$, then $A = BC_2$ so that (2) implies (3). Thus, (1), (2) and (3) have been shown to be equivalent.

Further consideration of the proofs of (2) implies (3) and of (3) implies (2) show us that (a) holds for the C_2 constructed in the preceding paragraph. Moreover, (b) holds for this C_2 because

$$\text{null } [C_2] = \text{range } [D]^{\perp} = \text{range } [A^*]^{\perp} = \text{null } [A]$$

and finally (c) holds, because

$$\text{range } [B^*]^{\perp} \subset \text{null } [D] = \text{range } [C_2]^{\perp}$$

which implies range $[C_2] \subset$range $[B^*]^-$. Lastly, we show that if E is an operator on $\mathfrak{H}$ for which $A = BE$ and range $[E] \subset$range $[B^*]^-$, then $E = C_2$. If $A = BE$, then $E^*B^* = A^* = C_2^*B^*$ so that $E^*f = C_2$ for $f \in$range $[B^*]^-$. If $f \in$range $[B^*]^{\perp}$, then $f \in$range $[E]^{\perp} =$null $[E^*]$ so that $E^*f = 0 = C_2^*f$. Thus, $E = C_2$ and the proof is complete.

We now state a version of Theorem 1 for unbounded operators. This result is less definitive than the preceding and it seems quite reasonable that additional assumptions about A and B would yield more information about C. In particular, it would be of interest to know what additional assumption in addition to range$[A] \subset$range $[B]$ is necessary to conclude that C is bounded.

THEOREM 2. *Let A and B be closed densely defined operators on $\mathfrak{H}$.*
(1) *If $AA^* \leqq BB^*$,*[1] *then there exists a contraction C so that $A \subset BC$.*

[1] The statement $AA^* \leq BB^*$ is assumed to mean that $D_{BB^*} \subset D_{AA^*}$ and for $f \in D_{BB^*}$ we have $(AA^*f, f) \leq (BB^*f, f)$.

(2) *If C is an operator so that $A \subset BC$, then* range $[A] \subset$ range $[B]$.

(3) *If* range $[A] \subset$ range $[B]$, *then there exists a densely defined operator C so that $A = BC$ and a number $M \geqq 0$ so that $\|Cf\|^2 \leqq M\{\|f\|^2 + \|Af\|^2\}$ for $f \in D_C$. Moreover, if A is bounded, then C is bounded, and if B is bounded, then C is closed.*

PROOF. (1) We define the operator C^* as before by setting $C^*B^*f = A^*f$ for $f \in D_{BB^*}$. Then C^* is a contraction because

$$\|C^*B^*f\|^2 = \|A^*f\|^2 = (AA^*f, f) \leqq (BB^*f, f) = \|B^*f\|^2.$$

If we extend C^* to range $[B^*]^-$ and define it to be zero on range $[B^*]^\perp$, then $C^*B^* \subset A^*$ so that $A \subset BC$.

(2) If $A \subset BC$, then range $[A] \subset$ range $[B]$.

(3) If range $[A] \subset$ range $[B]$, then we proceed as in the proof of Theorem 1 after making the observation that the null space of a closed operator is closed. Thus C is defined on D_A so that $A = BC$ and the closed graph theorem can be applied to C considered as an operator the graph of A to $\mathfrak{H}$. From this it follows that there exists $M \geqq 0$ so that $\|Cf\|^2 \leqq M(\|f\|^2 + \|Af\|^2)$ for $f \in D_A$. If A is bounded, then C is bounded and can be extended to all $\mathfrak{H}$. If B is bounded and the sequence $\{[f_n, Cf_n]\}_{n=1}^\infty$ converges to $[f, g]$, then the sequence $\{[f_n, BCf_n]\}_{n=1}^\infty$ converges to $[f, Bg]$. But $BCf_n = Af_n$ so that $\{[f_n, Af_n]\}_{n=1}^\infty$ converges to $[f, Bg]$. But A is closed so that $Bg = Af$ or $Cf = g$ and we see that C is closed.

Added in proof. We remark that two easy generalizations of the results of this paper are possible. Firstly, if we consider operators A and B with domains equal to the Hilbert spaces $\mathfrak{H}_1$ and $\mathfrak{H}_2$, respectively, but having common range $\mathfrak{K}$, then we need only modify the statement of Theorem 1 so that the operator C is now defined from $\mathfrak{H}_1$ to $\mathfrak{H}_2$ to obtain parallel results for this case. The proof is exactly the same.

Secondly, the equivalence of statements (1) and (3) of Theorem 1 persists if A and B are operators between Banach spaces. Similarly, the last statement of Theorem 3 is valid for closed operators between Banach spaces. Again the proofs are the same.

REFERENCE

1. N. I. Achieser and I. M. Glasman, *Theorie der Linearen Operatoren im Hilbert-Räum*, Akademie-Verlag, Berlin, 1960.

UNIVERSITY OF MICHIGAN