ORIENTED MANIFOLDS FIBERED OVER THE CIRCLE

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This note is concerned with the following problem: Which classes in the oriented cobordism ring Ω_* can be represented by a manifold fibered differentiably over the circle, and can one give a constructive method for obtaining such representatives? (For the unoriented case see [2]). Many examples are known of classes which can be represented by manifolds fibered over the 2-sphere. In §1 a construction is shown which enables one, if given an oriented manifold E fibered over the n-sphere where n is positive with fiber N, to obtain an oriented manifold fibered over the circle with fiber $S^{n-1}\times N$ representing the same oriented cobordism class as E. In §2 a homomorphism is defined from $\text{Diff}_+ (M)$, the group of orientation preserving diffeomorphisms of a closed oriented n-manifold M onto itself, into Ω_{n+1}, the $(n+1)$-dimensional oriented cobordism group. This is used to show that if a closed oriented manifold is fibered over the circle with a compact Lie group as the group of the fibration, then the manifold represents a torsion element in Ω_*. These results rely on a geometric construction described in §1 by which two manifolds with diffeomorphic boundaries are fitted together to form a closed manifold.

Geometric constructions. Throughout this note all manifolds will be compact C^∞-differentiable, but not necessarily connected; all diffeomorphisms will be C^∞-diffeomorphisms. Given an oriented manifold whose boundary is the disjoint union ("+") of closed manifolds N_1 and N_2, there are the induced orientation classes $\iota_1\in H_{n-1}(N_1; Z)$ and $\iota_2\in H_{n-1}(N_2; Z)$. Let f be a diffeomorphism of N_1 onto N_2 with $f_*(\iota_1)=-\iota_2$ (denote this by $f\colon N_1\to -N_2$, and in general let $-M$ be the manifold M with opposite orientation). The identification space $M(f)$ formed from M by identifying x with $f(x)$ for $x\in N_1$ can be made into a closed oriented n-manifold. The differentiable structure on $M(f)$ is as follows (see [4]): First the identification map $i=M\to M(f)$ is to be a diffeomorphism on the interior of M. There is a neighborhood U_1 of N_1 in M and a diffeomorphism p_1 of U_1 onto $N_1\times [0, 1)$. Likewise there is a neighborhood U_2 of N_2 in M, and a diffeomorphism p_2 of U_2 onto $N_2\times [0, 1)$. U_1 and U_2 can be chosen disjoint. Let $U=i(U_1)\cup i(U_2)$ in $M(f)$, and let $p\colon U\to N_2\times(-1, 1)$ be the homeomorphism induced by p_1, f, and p_2. The differentiable structure is

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then obtained by requiring p to be a diffeomorphism. In this building of manifolds from pieces with diffeomorphic boundaries, the diffeomorphism f is of course important, but it is independent of the choice of U_1, p_1 and U_2, p_2 (see [4], p. 63).

For a closed oriented n-manifold B, let $[B]$ be the cobordism class containing B in Ω_n.

THEOREM (1.1). *If M is an oriented n-manifold whose boundary is N_1+N_2, and f, $g: N_1 \to -N_2$ are diffeomorphisms onto, then in Ω_n*

$$[M(g)] = [M(f)] + [(N_1 \times I)(f^{-1}g)].$$

The last term on the right is the class of the manifold obtained from $N_1 \times I$, I the unit interval, by identifying $(x, 0)$ with $(f^{-1}g(x), 1)$.

PROOF. By "straightening the angle" [1], $M \times I$ can be made into an oriented $(n+1)$-manifold whose boundary is $-M\times\{0\}\cup \partial M\times I\cup M\times\{1\}$. In $M\times I$ identify (x, t) with $(g(x), t)$ for $x\in N_1$ and $0\leqq t\leqq 1/4$, and (x, t) with $(f(x), t)$ for $x\in N_1$ and $3/4\leqq t\leqq 1$. The resulting space can be made into an oriented $(n+1)$-manifold whose boundary is $-M(g)$ on the bottom, $M(f)$ on the top, and on the side the following manifold: It is formed from $N_1\times[1/4, 3/4]+N_2\times[1/4, 3/4]$ by joining $(x, 1/4)$ to $(g(x), 1/4)$ and $(x, 3/4)$ with $(f(x), 3/4)$ for $x\in N_1$. But this is diffeomorphic to $N_1\times I$ with $(x, 0)$ identified with $(f^{-1}g(x), 1)$ via the diffeomorphism $(x, t)\to(x, t-1/4)$ for $x\in N_1$ and $(y, t)\to(f^{-1}(y), 5/4-t)$ for $y\in N_2$. Thus the conclusion of the theorem follows. Observe that $(N_1\times I)(f^{-1}g)$ is a bundle over the circle whose fiber is N_1 and whose group is the cyclic subgroup generated by $f^{-1}g$ in $\mathrm{Diff}_+(N_1)$, the group of orientation preserving diffeomorphisms of N_1 onto itself. In fact it can be described as follows: Let R be the reals and Z the integers. Let Z act on $N_1\times R$ by $(x, t)\cdot n=((f^{-1}g)^n(x), t+n)$, then $(N_1\times I)(f^{-1}g)$ is the bundle space of the Z-bundle $N_1\times R/Z\to R/Z\approx S^1$.

The above construction is used to show the following:

THEOREM (1.2). *If a class in Ω_* can be represented by a manifold fibered differentiably over the n-sphere where n is positive, then it can be represented by a manifold fibered differentiably over the circle.*

PROOF. Let $\{E, S^n, N, \pi, G\}$ be a fiber bundle over the n-sphere S^n with total space the oriented $(n+m)$-manifold E, fiber the closed oriented m-manifold N, projection map $\pi: E\to S^n$ and group G a subgroup of $\mathrm{Diff}_+(N)$. Represent E in the following manner: Split S^n into two hemispheres D_+ and D_- which intersect in S^{n-1}. These being n-cells the bundle is trivial over each. Thus there is a map $x\to g_x$ of S^{n-1} into G, so that E will be the space obtained from the disjoint union of two copies of $I^n \times N$ with opposite orientation by identifying (x, y) in one with $(x, g_x(y))$ in the other for $x \in \partial I^n \approx S^{n-1}$. Let $(I^n \times N \times I)_1 + (-I^n \times N \times I)_2$ be two disjoint copies of $I^n \times N \times I$ with opposite orientation. Identify $(x, y, t)_1$ with $(x, y, t)_2$ for $x \in \partial I^n$, $y \in N$, and $0 \leqq t \leqq 1/4$. This will give $S^n \times N$ on the bottom. Join $(x, y, t)_1$ with $(x, g_x(y), t)_2$ for $x \in \partial I^n$, $y \in N$, and $3/4 \leqq t \leqq 1$. This will give E on the top. Since $[S^n \times N] = 0$ in Ω_{n+m}, E is cobordant to the remaining part of the boundary. But as before this is $(S^{n-1} \times N \times I)(F)$ where $F(x, y) = (x, g_x(y))$ is in $\mathrm{Diff}_+ (S^{n-1} \times N)$. This is a bundle over the circle whose fiber is $S^{n-1} \times N$ and whose group is the cyclic subgroup generated by F in $\mathrm{Diff}_+ (S^{n-1} \times N)$.

The homomorphism β. As before if M is a closed oriented n-manifold and $f \in \mathrm{Diff}_+ (M)$, let $(M \times I)(f)$ be the manifold obtained by identifying $(x, 0)$ with $(f(x), 1)$. Define the function

$$\beta \colon \mathrm{Diff}_+ (M) \to \Omega_{n+1} \text{ by } \beta(f) = [(M \times I)(f)].$$

THEOREM (2.1). *β is a homomorphism.*

PROOF. Let $id \in \mathrm{Diff}_+ (M)$ be the identity diffeomorphism. If $f \in D\mathrm{iff}_+ (M)$, then by (1.1)

$$[(M \times I)(id)] = [(M \times I)(f)] + [(M \times I)(f^{-1})]$$

but

$$[(M \times I)(id)] = [M \times S^1] = 0$$

so

$$\beta(f) = -\beta(f^{-1}). \tag{1}$$

If f and g are in $\mathrm{Diff}_+ (M)$

$$[(M \times I)(g)] = [(M \times I)(f^{-1})] + [(M \times I)(fg)]$$

so

$$\beta(g) = \beta(f^{-1}) + \beta(fg)$$

or by (1)

$$\beta(f) + \beta(g) = \beta(fg).$$

Therefore β is a homomorphism.

As an example where β is nontrivial, let $M = CP(2)$, 2-dimensional complex projective space, four real dimensions. A point $z \in CP(2)$ is expressed in homogeneous coordinates $z = [z_1, z_2, z_3]$ where the z_i's are complex numbers. Let f be conjugation $f(z) = [\bar{z}_1, \bar{z}_2, \bar{z}_3]$. Then Thom has shown [**3**, p. 82] that $\beta(f)$ is the generator of $\Omega_5 \cong Z_2$.

THEOREM (2.2). *If f, $g \in Diff_+(M)$ are differentiably isotopic, then $\beta(f) = \beta(g)$.*

PROOF. f and g being isotopic, there exists a differentiable function $F: M \times I \to M$ with $F(x, 0) = g(x)$, $F(x, 1) = f(x)$ and $F(x, t) = f_t(x)$ a diffeomorphism for each t. $M \times I \times I$ is an $(n+2)$-manifold whose boundary is

$$M \times I \times \{0\} \cup M \times \{0\} \times I \cup M \times \{1\} \times I \cup M \times I \times \{1\}.$$

In this boundary identify $(x, 0, t)$ with $(f_t(x), 1, t)$.

THEOREM (2.3). *If $\{E, S^1, N, \pi, G\}$ is a fiber bundle over the circle with E an oriented n-manifold, N an oriented $(n-1)$-manifold and G a compact Lie group acting on N as a group of orientation preserving diffeomorphisms, then E represents a torsion element in Ω_*.*

PROOF. The bundle is equivalent to $(N \times I)(g)$ for some diffeomorphism $g \in G$. Since G is compact, for some n, g^n is in the identity component of G. So there exists a one parameter subgroup of G through g^n. This gives a differentiable map $F: I \to G$ with $F(0)$ the identity of G and $F(1) = g^n$. Then the composite map $I \times N \to G \times N \to N$ is an isotopy connecting g^n with the identity. So by (2.2)

$$0 = \beta(id) = \beta(f^n) = n\beta(f).$$

But $\beta(f) = [E]$ so $n[E] = 0$ which proves the theorem.

Given an oriented manifold N on which S^1 acts as a group of orientation preserving diffeomorphisms, there is the bundle over the 2-sphere with fiber N obtained by joining two copies of $I \times N$ by means of $(x, y)_1 \sim (x, x \cdot y)_2$ for $x \in \partial I \approx S^1$ and $y \in N$. By (1.2) this is cobordant to the bundle over the circle $(S^1 \times N \times R)/Z \to R/Z$ where the action of Z is $(x, y, t) \cdot n = (x, x^n \cdot y, t + n)$. In [1, p. 120] examples are constructed of oriented manifolds fibered over the 2-sphere whose classes are of infinite order in Ω_*. Thus these classes can be represented by manifolds fibered over the circle.

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