ON k-SPACES AND FUNCTION SPACES

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Let F be a family of continuous functions from a topological space X to a topological space Y. Denote by $\mathcal{C}$ the compact-open topology on F. In this paper we consider the problem of when the product of two topological spaces is a k-space and the related question: When does compactness of $(F, \mathcal{C})$ or, more general, local compactness imply that $\mathcal{C}$ is jointly continuous? It is shown that the product of a locally compact Hausdorff space with a Hausdorff k-space is a k-space. This is used to prove that $\mathcal{C}$ is jointly continuous when $(F, \mathcal{C})$ is locally compact and X is a Hausdorff k-space. These results are combined with a theorem of R. Arens [1] to construct an example of two k-spaces whose product is not a k-space. We also prove a generalization of the Ascoli Theorem 7.21 [2, Theorem 21, Chapter 7]. In a remark following this theorem Kelley points out that it can be extended to k-spaces by weakening the condition on even continuity. We show that the theorem holds for Hausdorff k-spaces without alteration, Theorem 4. The same remark holds for [2, Theorem 7.17].

THEOREM 1. *If X is a locally compact Hausdorff space and Y is a Hausdorff k-space, then $X \times Y$ is a k-space.*

PROOF. Let C be a subset of $X \times Y$ which intersects every compact set in a closed set. Let $(x, y) \in \overline{C}$, V be a compact neighborhood of x and U any compact neighborhood of x contained in V. Define $T = \pi_1(C \cap (V \times \{y\}))$ and $S = \pi_2(C \cap (U \times Y))$ where π_1, π_2 are the projections into X, Y respectively. If A is any compact subset of Y, then $S \cap A = \pi_2(C \cap (U \times A))$. Thus, S is closed since Y is a k-space and Hausdorff. If W is a neighborhood of y, then $C \cap (U \times W) \neq \emptyset$ and $S \cap W = \pi_2(C \cap (U \times W)) \neq \emptyset$. Thus, it follows that $y \in S$ and $U \cap T \neq \emptyset$. Since T is closed and U was an arbitrary compact neighborhood of x contained in V, $x \in T$ and hence $(x, y) \in C$. The proof is complete.

LEMMA. *Let X and Y be Hausdorff spaces, $F \subset C(X, Y)$ and let τ be a topology on F which contains $\mathcal{C}$ and such that $(F, \tau) \times X$ is a k-space. Then τ is jointly continuous for F.*

PROOF. Let C be a closed subset of Y and K a compact subset of

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$(F, \tau)\times X$. Let $M=K\cap P^{-1}(C)$ and $(f, x)\notin M$, where P is the evaluation mapping of $F\times X$ into Y. If $(f, x)\notin K$, then obviously $(f, x)\notin \overline{M}$. Suppose $(f, x)\in K$ and $(f, x)\notin P^{-1}(C)$. Let $U=Y-C$ and K_X be the projection of K into X. There is a compact neighborhood N of x relative to K_X such that $f(N)\subset U$ and $P([N, U]\times N)\subset U$, where $[N, U]=\{g\in F\mid g(N)\subset U\}\in \mathcal{C}\subset\tau$. Thus, $([N, U]\times N)\cap P^{-1}(C)=\emptyset$. It follows that (f, x) is not in the closure, relative to $(F, \tau)\times K_X$, of M. But, since $M\subset K\subset F\times K_X$, we have $(f, x)\notin\overline{M}$. Since $F\times X$ is a k-space, $P^{-1}(C)$ is closed and the proof is complete.

The product of two k-spaces need not be a k-space. As a matter of fact the example below shows that, even if one of the spaces is metric, the product need not be a k-space. We have not been able to settle the question whether the product of two hereditary k-spaces is a k-space.

Example. Let X be the dual space of an infinite dimensional Fréchet space with the compact-open topology. As Warner [**4**, p. 267] points out, X is a hemicompact k-space which is not locally compact. Now $F=C(X, [0, 1])$ with the compact-open topology is metrizable, [**4**, Theorem 2]. Suppose $X\times F$ is a k-space. Then, by the Lemma the compact-open topology is jointly continuous. Since X is completely regular, it follows from [**1**, Theorem 3] that X is locally compact which is a contradiction, and consequently the product $X\times F$ is not a k-space. It follows from [**3**, Proposition 4] that $X\times F$ is paracompact.

The following is a generalization of (b) [**1**, p. 486]. (Cf. [**4**, Theorems 13 and 17].)

Remark. If X is completely regular and $X\times C(X, [0, 1])$ is a k-space, where $C(X, [0, 1])$ has the compact-open topology, then X is locally compact.

Proof. The proof is immediate using [**1**, Theorem 3] and the Lemma.

From Theorem 1 and the Lemma we have,

Theorem 2. *If $(F, \mathcal{C})$ is locally compact, X a Hausdorff k-space and Y Hausdorff, then $\mathcal{C}$ is jointly continuous for F.*

Using Theorem 2, we now have generalizations of [**2**, Theorem 7.17 and Theorem 7.21]. The proofs are the same as Kelley's by virtue of Theorem 2.

Theorem 3. *Let X be a Hausdorff k-space and Y a Hausdorff uniform space. Let $F\subset C(X, Y)$. Then $(F, \mathcal{C})$ is compact if and only if*

(a) *$(F, \mathcal{C})$ is closed.*

(b) *$F(x)$ has compact closure for each $x \in X$.*

(c) *F is equicontinuous.*

THEOREM 4. *Let X be a Hausdorff k-space and Y a regular Hausdorff space. Let $F \subset C(X, Y)$. Then $(F, \mathcal{C})$ is compact if and only if*

(a) *$(F, \mathcal{C})$ is closed.*

(b) *$F(x)$ has compact closure for each $x \in X$.*

(c) *F is evenly continuous.*

Added in proof. T. S. Wu has referred us to a paper of D. E. Cohen (*Spaces with weak topology*, Quart. J. Math. Oxford Ser. **5** (1954), 77–80) in which a theorem of J. H. C. Whitehead was used to obtain Theorem 1. The proof here is direct and simpler.

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