ON \( k \)-SPACES AND FUNCTION SPACES

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Let \( F \) be a family of continuous functions from a topological space \( X \) to a topological space \( Y \). Denote by \( \mathcal{C} \) the compact-open topology on \( F \). In this paper we consider the problem of when the product of two topological spaces is a \( k \)-space and the related question: When does compactness of \((F, \mathcal{C})\) or, more general, local compactness imply that \( \mathcal{C} \) is jointly continuous? It is shown that the product of a locally compact Hausdorff space with a Hausdorff \( k \)-space is a \( k \)-space. This is used to prove that \( \mathcal{C} \) is jointly continuous when \((F, \mathcal{C})\) is locally compact and \( X \) is a Hausdorff \( k \)-space. These results are combined with a theorem of R. Arens [1] to construct an example of two \( k \)-spaces whose product is not a \( k \)-space. We also prove a generalization of the Ascoli Theorem 7.21 [2, Theorem 21, Chapter 7]. In a remark following this theorem Kelley points out that it can be extended to \( k \)-spaces by weakening the condition on even continuity. We show that the theorem holds for Hausdorff \( k \)-spaces without alteration, Theorem 4. The same remark holds for [2, Theorem 7.17].

THEOREM 1. If \( X \) is a locally compact Hausdorff space and \( Y \) is a Hausdorff \( k \)-space, then \( X \times Y \) is a \( k \)-space.

PROOF. Let \( C \) be a subset of \( X \times Y \) which intersects every compact set in a closed set. Let \((x, y)\) \( \in \overline{C} \), \( V \) be a compact neighborhood of \( x \) and \( U \) any compact neighborhood of \( x \) contained in \( V \). Define \( T = \pi_1(C \cap (V \times \{y\})) \) and \( S = \pi_2(C \cap (U \times Y)) \) where \( \pi_1, \pi_2 \) are the projections into \( X, Y \) respectively. If \( A \) is any compact subset of \( Y \), then \( S \cap A = \pi_2(C \cap (U \times A)) \). Thus, \( S \) is closed since \( Y \) is a \( k \)-space and Hausdorff. If \( W \) is a neighborhood of \( y \), then \( C \cap (U \times W) \neq \emptyset \) and \( S \cap W = \pi_2(C \cap (U \times W)) \neq \emptyset \). Thus, it follows that \( y \in S \) and \( U \cap T \neq \emptyset \). Since \( T \) is closed and \( U \) was an arbitrary compact neighborhood of \( x \) contained in \( V \), \( x \in T \) and hence \((x, y) \in C \). The proof is complete.

LEMMA. Let \( X \) and \( Y \) be Hausdorff spaces, \( F \subseteq C(X, Y) \) and let \( \tau \) be a topology on \( F \) which contains \( \mathcal{C} \) and such that \((F, \tau) \times X \) is a \( k \)-space. Then \( \tau \) is jointly continuous for \( F \).

PROOF. Let \( C \) be a closed subset of \( Y \) and \( K \) a compact subset of

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Let \( M = KT \cap P^{-1}(C) \) and \((f, x) \in M\), where \( P \) is the evaluation mapping of \( F \times X \) into \( Y \). If \((f, x) \notin K\), then obviously \((f, x) \notin M\).

Suppose \((f, x) \in K\) and \((f, x) \in P^{-1}(C)\). Let \( U = Y - C \) and \( K_x \) be the projection of \( K \) into \( X \). There is a compact neighborhood \( N \) of \( x \) relative to \( K_x \) such that \( f(N) \subseteq U \) and \( P([N, U] \times N) \subseteq U \), where \([N, U] = \{g \in F | g(N) \subseteq U\}\). Thus, \(([N, U] \times N) \cap P^{-1}(C) = \emptyset\). It follows that \((f, x)\) is not in the closure, relative to \((F, \tau) \times K_x\), of \( M \). But, since \( M \subseteq K \subseteq F \times K_x \), we have \((f, x) \notin \overline{M}\). Since \( F \times X \) is a \( k \)-space, \( P^{-1}(C) \) is closed and the proof is complete.

The product of two \( k \)-spaces need not be a \( k \)-space. As a matter of fact the example below shows that, even if one of the spaces is metric, the product need not be a \( k \)-space. We have not been able to settle the question whether the product of two hereditary \( k \)-spaces is a \( k \)-space.

**Example.** Let \( X \) be the dual space of an infinite dimensional Fréchet space with the compact-open topology. As Warner [4, p. 267] points out, \( X \) is a hemicompact \( k \)-space which is not locally compact. Now \( F = C(X, [0, 1]) \) with the compact-open topology is metrizable, [4, Theorem 2]. Suppose \( X \times F \) is a \( k \)-space. Then, by the Lemma the compact-open topology is jointly continuous. Since \( X \) is completely regular, it follows from [1, Theorem 3] that \( X \) is locally compact which is a contradiction, and consequently the product \( X \times F \) is not a \( k \)-space. It follows from [3, Proposition 4] that \( X \times F \) is paracompact.

The following is a generalization of (b) [1, p. 486]. (Cf. [4, Theorems 13 and 17].)

**Remark.** If \( X \) is completely regular and \( X \times C(X, [0, 1]) \) is a \( k \)-space, where \( C(X, [0, 1]) \) has the compact-open topology, then \( X \) is locally compact.

**Proof.** The proof is immediate using [1, Theorem 3] and the Lemma.

From Theorem 1 and the Lemma we have,

**Theorem 2.** If \((F, \mathcal{E})\) is locally compact, \( X \) a Hausdorff \( k \)-space and \( Y \) Hausdorff, then \( \mathcal{E} \) is jointly continuous for \( F \).

Using Theorem 2, we now have generalizations of [2, Theorem 7.17 and Theorem 7.21]. The proofs are the same as Kelley's by virtue of Theorem 2.

**Theorem 3.** Let \( X \) be a Hausdorff \( k \)-space and \( Y \) a Hausdorff uniform space. Let \( F \subseteq C(X, Y) \). Then \((F, \mathcal{E})\) is compact if and only if

(a) \((F, \mathcal{E})\) is closed.
(b) $F(x)$ has compact closure for each $x \in X$.
(c) $F$ is equicontinuous.

**Theorem 4.** Let $X$ be a Hausdorff $k$-space and $Y$ a regular Hausdorff space. Let $F \subseteq C(X, Y)$. Then $(F, \mathcal{C})$ is compact if and only if
(a) $(F, \mathcal{C})$ is closed.
(b) $F(x)$ has compact closure for each $x \in X$.
(c) $F$ is evenly continuous.

*Added in proof.* T. S. Wu has referred us to a paper of D. E. Cohen (Spaces with weak topology, Quart. J. Math. Oxford Ser. 5 (1954), 77–80) in which a theorem of J. H. C. Whitehead was used to obtain Theorem 1. The proof here is direct and simpler.

**References**


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