ON THE SQUARES OF ORIENTED MANIFOLDS

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1. Introduction. The object of this paper is to give another proof of the Milnor conjecture [2]:

**Theorem 1.** The square of an oriented manifold is unoriented cobordant to a Spin manifold.

This result was proved by Anderson [1]. The proof given here is patterned directly on the method used by Wall [4] in the determination of the oriented cobordism ring. This method provides additional information, and the principal result will be:

**Theorem 2.** Let $S'$ denote the subset of $H^*(BO, \mathbb{Z}_2)$ consisting of all classes $w_{2i+1}$. Let $S$ be either $S' \cup \{w_2\}$ or $S' \cup \{w_2^2\}$. Let $M$ be a manifold such that every Stiefel-Whitney number of $M$ divisible by a class of $S$ is zero. Then:

(a) For $S = S' \cup \{w_2^2\}$, $M$ is unoriented cobordant to a complex manifold $M'$ with $w_2^2$ zero.

(b) For $S = S' \cup \{w_2\}$, $M$ is unoriented cobordant to the sum of an SU manifold and a polynomial in the quaternionic projective spaces $QP(2n)$.

Theorem 1 is a direct consequence of the case $S = S' \cup \{w_2\}$, and one has the improved result:

**Corollary.** If $M$ is an oriented manifold with dimension not divisible by 4, then $M \times M$ is unoriented cobordant to an SU manifold.

2. Proofs of the results. First consider the case $S = S' \cup \{w_2^2\}$ and suppose $M$ has all Stiefel-Whitney numbers divisible by elements of $S$ zero.

By Milnor [2], $M$ is cobordant to a complex manifold $N$, since all numbers of $M$ divisible by elements of $S'$ are zero. Then let $M' \subset N \times CP(1)$ be a submanifold dual to $c_1(N) + \alpha$, $\alpha \in H^2(CP(1), \mathbb{Z})$ being the usual generator. The total Chern class of $M'$ is the restriction to $M'$ of

\[
\frac{c(N) \cdot (1 + \alpha)^2}{1 + \alpha + c_1(N)} = 1 + \alpha + (c_2(N) + \alpha c_1(N)) + \cdots ,
\]

Received by the editors August 8, 1965.

1 The author is indebted to the National Science Foundation for financial support during this work.
so $c_2'(M')$ is the restriction to $M'$ of $\alpha^2$, hence is zero. Reducing mod 2,
\[ w(M') = \frac{w(N)}{1 + \alpha + w_2(N)}, \]
and $M'$ is dual to $\alpha + w_2(N)$. Then for any $\omega = (i_1, \ldots, i_r)$, \(\sum i_s = \dim M\), \(w_\omega = w_{i_1} \cdots w_{i_r}\), one has
\[ w_\omega(M') = w_\omega(N) + (\alpha + w_2(N))v_\omega, \]
where $v_\omega$ is a polynomial in $\alpha$ and the $w_i(N)$. Multiplying by $(\alpha + w_2(N))$ and evaluating on $N \times \text{CP}(1)$ gives
\[ w_\omega[M'] = \{ w_\omega(N) \cdot (\alpha + w_2(N)) + (\alpha + w_2(N))^2v_\omega \}[N \times \text{CP}(1)], \]
and the second term is zero since every number of $N$ divisible by $w_2^2$ is zero. Then $w_\omega[M'] = w_\omega[N]$ for all $\omega$, and by Thom [3], $M'$ and $N$ are cobordant since they have the same Stiefel-Whitney numbers. This proves part (a) of the theorem.

*Note.* Using the same process with $\mathbb{RP}(1)$ and its class $\alpha$ gives a direct proof of Wall's result [4] that if $M$ has all numbers divisible by $w_2^2$ zero, then $M$ is cobordant to $M'$ with $w_2^2 = 0$.

Let $W_\ast$ denote Wall's ring of cobordism classes for which numbers divisible by $w_2^2$ are zero. Since
\[ w_{i_1} \cdots w_{i_r}[N \times N] = \begin{cases} 0 & \text{if any } i_s \text{ is odd,} \\ w_{j_1} \cdots w_{j_r}[N] & \text{if } i_s = 2j_s \text{ for all } s, \end{cases} \]
one has $W_\ast^2 = \{ \alpha^2 | \alpha \in W_\ast \}$ is the set of cobordism classes with numbers divisible by elements of $S^2 \cup \{ w_2^2 \}$ zero.

Let $\partial_1 : W_\ast \to W_\ast$ be the derivation defined by sending the cobordism class of $M$ to the class of a submanifold dual to $w_1$. If $\partial_2 : W_\ast^2 \to W_\ast^2 : \alpha^2 \mapsto (\partial_1 \alpha)^2$, then $\partial_2$ is a derivation and sends the cobordism class of $M$ to the class of a manifold $N$ such that $w_\omega[N] = w_2 \cdot w_\omega[M]$ for all $\omega$ (by the formula for $w_\omega[A \times A]$).

Now if $\alpha^2 \in W_\ast^2$, let $M$ be a complex manifold with $c_2^2 = 0$ belonging to $\alpha^2$. Let $N$ be a submanifold dual to $c_1$ in $M$. Then $w_\omega[N] = w_2 \cdot w_\omega[M]$ for all $\omega$, so $N$ belongs to $(\partial_1 \alpha)^2 = \partial_2(\alpha^2)$, and $N$ is an SU manifold. Thus $\text{Im } \partial_2$ consists entirely of cobordism classes of SU manifolds.

By Wall [4], $\ker \partial_1/\text{Im } \partial_1$ is the $\mathbb{Z}_2$ polynomial algebra on the images of the cobordism classes of the complex projective spaces $\text{CP}(2n)$, so $\ker \partial_2/\text{Im } \partial_2$ is the $\mathbb{Z}_2$ polynomial algebra on the images
of the cobordism classes of the \( \text{CP}(2n)^2 \) (which is the same as that of \( \text{QP}(2n) \)).

Now let \( M \) be any manifold all of whose Stiefel-Whitney numbers divisible by elements of \( S' \cup \{ w_2 \} \) are zero. Then the cobordism class \( \alpha \) of \( M \) belongs to \( \ker \partial_2 \). Thus there is a polynomial \( P \) in the \( \text{QP}(2n) \) with cobordism class \( \gamma \) for which \( \alpha - \gamma \in \text{Im } \partial_2 \). Hence there is an SU manifold \( N \) such that \( M \) is cobordant to \( N + P \), proving part (b) of Theorem 2.

Note. \( \ker \partial_2 = \{ \alpha^2 | \alpha \in \ker \partial_1 \} \) is precisely the cobordism classes of squares of oriented manifolds.

REFERENCES


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