LOCALLY CONNECTED 2-CELL AND 2-SPHERE-LIKE CONTINUA

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1. **Introduction.** The work described in this paper began with the conjecture that every locally connected 2-cell-like continuum can be embedded in the plane. The solution of this conjecture given in §2 also led to a proof that the continua considered all have the fixed point property. It became clear that the methods used applied as well to 2-sphere-like continua. A characterization of locally connected 2-sphere-like continua is given in the third section.

The previous study of 2-cell-like and 2-sphere-like continua has been connected with inverse limits. Fort and Segal [**5**] gave a condition ensuring that a locally connected 2-cell-like continuum is a 2-cell. Segal [**8**] showed that every cyclic locally connected 2-cell-like continuum is a 2-cell. Mardešić and Segal [**7**] showed that a locally connected 2-sphere-like continuum is a 2-sphere if either cyclic of 2-dimensional. Fort and Segal [**5**] showed that each locally connected 2-sphere-like continuum is a cactoid. The last three of these results are consequences of the theorems of this paper. The use of inverse limits will be avoided here by using the theory of locally connected continua.

The cyclic element theory of Whyburn [**10**] and a consequence of Claytor's theorems [**2**], [**3**] characterizing planar locally connected continua which was developed by Segal [**9**] to apply to quasi-embeddability are the principal machinery needed. All terms not defined here are as in [**10**]. All spaces are assumed to be metric. The closure of a set A will be denoted A^-. A function f whose domain is a metric space X is an ϵ-map if $f^{-1}f(x)$ has diameter less than ϵ for every x in X. A continuum M is said to be 2-cell-like (2-sphere-like) [**7**] if for each positive number ϵ there is an ϵ-map of M onto a 2-cell (2-sphere). It is easy to see that these are topological properties.

The author would like to thank Jack Segal for suggesting the reference [**6**]. Jack Segal pointed out that the conclusion that each locally connected 2-sphere-like continuum is a dendrite or a 2-sphere is an immediate consequence or Theorem 1 of [**7**] and Theorems 5 and 6 of [**4**]. The referee pointed out that with Theorem 3 of this paper one could also give a proof of Theorem 7 using inverse limits. Several suggestions made by the referee contributed considerably to the clarity of this paper.

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2. **Locally connected 2-cell-like continua.** To show that a 2-cell-like locally connected continuum is planar we need only show that it does not contain a secondary skew curve [**9**, Theorem 3.6]. The proof of the embedding theorem is broken into a sequence of smaller theorems which are easier to handle.

THEOREM 1 [**6**]. *If for each positive number ϵ there is an ϵ-map from the continuum M onto a unicoherent continuum, then M is unicoherent.*

THEOREM 2 [**4**]. *Let J be a simple closed curve which is the union of four arcs $A(1)$, $A(2)$, $A(3)$, and $A(4)$ having at most endpoints in common such that the pairs $A(1)$ and $A(3)$, and $A(2)$ and $A(4)$ are disjoint. If X is a metric space which is the union of $X(1)$, $X(2)$, $X(3)$ and $X(4)$ where for each n $X(n)$ is closed in X, $X(n)$ contains $A(n)$ and the pairs $X(1)$ and $X(3)$, and $X(2)$ and $X(4)$ are disjoint, then there is a retraction of X to J.*

Theorem 2 was an essential lemma in M. K. Fort's proof that 2-cells are not torus-like [**4**]. His proof inspired the proof of the next theorem, which also makes use of Theorem 2, although the details differ.

THEOREM 3. *Suppose M is a continuum which is the union of an arc A and a disc D such that the intersection of D and A is a single point which is an endpoint of A and an interior point of D. Then there is a positive number ϵ such that there is no ϵ-map of M into either the plane or a 2-sphere.*

PROOF. We may assume D is the disc $\{(x, y, 0) \mid x^2+y^2 \leqq 1\}$ and A is the arc $\{(0, 0, z) \mid 0 \leqq z \leqq 1\}$. It will be sufficient to show that there is no (1/8)-map of M into a 2-sphere S. Suppose f is a (1/8)-map of M into S. There are disjoint simple closed curves $J(2)$, $J(3)$ and $J(4)$ such that each $J(n)$ is the union of four arcs $J(n, k)$, $k=1, 2, 3, 4$, having at most endpoints in common, such that each $J(n, k)$ is a subset of the image of the subarc of M which is on the circle in D of radius $n/4$ and center at the origin and is in the kth quadrant of the plane $z=0$. By Theorem 2 the image of the subset $K=\{(x, y, 0) \mid 1/4 \leqq x^2+y^2 \leqq 1\}$ of M can be retracted to $J(n)$, $n=2, 3, 4$. Since $J(4)$ does not intersect the image of the connected set $(M-K)^-$, it cannot be true that $f(M)$ covers both complementary domains of $J(4)$. So $f(M)$ is a proper subset of S. The complementary domain of $J(4)$ in S that contains $f(M-K)$ also contains $J(3)$ and $J(2)$. Moreover, that complementary domain is not a subset of the image of the disc D, since $f(0, 0, 1)$ is not in $f(D)$. There are four disjoint arcs $B(1)$, $B(2)$,

$B(3)$ and $B(4)$ from $J(2)$ to $J(4)$ contained in $f(M)$ which are, respectively, subsets of the images of the closed $(1/8)$-neighborhoods of the straight line intervals from $(1/2, 0, 0)$ to $(1, 0, 0)$, from $(0, 1/2, 0)$ to $(0, 1, 0)$, from $(-1/2, 0, 0)$ to $(-1, 0, 0)$, and from $(0, -1/2, 0)$ to $(0, -1, 0)$. Since there is an arc from $J(3)$ to the interior of $B(1)$ that misses both $J(2)$ and $J(4)$, $J(3)$ is between $J(2)$ and $J(4)$. The arc $J(3, 1)$ is in the complementary domain containing $B(1)$ and $B(2)$ of the simple closed curve which is the union of $B(3)$, $B(4)$ and the arcs of $J(2)$ and $J(4)$ from $B(3)$ to $B(4)$ which miss $B(1)$ and $B(2)$. Similar statements place the arcs $J(3, 2)$, $J(3, 3)$ and $J(3, 4)$ relative to the arcs $B(n)$, $n=1, 2, 3, 4$. There is an extension g of f' to all of M which is a homeomorphism of the boundary of D, where f' is f restricted to the complement in M of $\{(x, y, 0) \mid x^2+y^2>(3/4)^2\}$. This extension can be chosen so that $g(K)$ misses $g(A)$. But then there is a map of a disc into a proper subset of itself which leaves the boundary fixed, which is impossible.

The next theorem will be used in the characterization of the true cyclic elements of locally connected 2-cell-like continua.

THEOREM 4. *If M is a nondegenerate locally connected, cyclic and unicoherent plane continuum, then M is a 2-cell.*

PROOF. The boundary of each complementary domain of M is a simple closed curve. ((2.5), Chapter 6 of [**10**].) Suppose M is not the 2-cell bounded by the boundary J of the unbounded complementary domain of M. Then there is a simple closed curve K contained in M which is not J and such that no point of M is interior to K. If K does not intersect J then there are disjoint arcs A and B from J to K. ((9.1), Chapter 4 of [**10**].) But then M is the union of two subcontinua whose intersection is the union of A and B, which contradicts the assumption that M is unicoherent. If there is a point p in the intersection of J and K, there is an arc A in M from J to K missing p. But then M is the union of two subcontinua whose intersection is the union of $\{p\}$ and A, again contradicting the assumption that M is unicoherent.

The secondary skew curves V and W (called K_3 and K_4 by Segal [**9**]) can be described as follows. Let X be the subset of the plane which is the union of the unit circle and the straight line intervals from $(0, 1)$ to $(0, -1)$ and from $(-1, 0)$ to $(0, 0)$. Let Y be the union of X and an arc from $(0, -1)$ to $(0, 1)$ which meets X only in its endpoints and one point of the open interval from $(-1, 0)$ to $(0, 0)$. Then V is the union of a sequence $A(0), A(1), \cdots$ of disjoint arcs and a sequence of disjoint continua $V(1), V(2), \cdots$ homeomorphic

to X, such that the $V(n)$ converge to one endpoint of $A(0)$ and $A(n)$ runs between points of $V(n)$ and $V(n+1)$ corresponding to $(0, 1/2)$ and $(-2^{-1/2}, -2^{-1/2})$ for each positive integer n. The continuum W is the union of a sequence $B(0), B(1), \cdots$ of disjoint arcs and a sequence $W(1), W(2), \cdots$ of disjoint continua homeomorphic to Y such that the $W(n)$ converge to one end-point of $B(0)$ and $B(n)$ runs between points of $W(n)$ and $W(n+1)$ corresponding to $(0, 0)$ and $(-1, 0)$ for each positive integer n.

THEOREM 5. *Every true cyclic element of a locally connected 2-cell-like continuum M is a 2-cell.*

PROOF. Necessarily a true cyclic element C of M is nondegenerate, cyclic, locally connected, unicoherent and contains no primitive skew curve since M is locally connected, unicoherent and contains no primitive skew curve. Therefore C can be embedded in the plane [**2**]. By Theorem 4, C is a 2-cell.

COROLLARY 6. *Every locally connected 2-cell-like continuum has the fixed point property.*

PROOF. All 2-cells have the fixed point property, and the fixed point property is cyclically extensible ((3.2), Chapter 12 of [**10**]) for locally connected continua.

It is a consequence of the embedding theorem that locally connected 2-cell-like continua are retracts of 2-cells since they are base sets (see after Theorem 7 and (3.31), Chapter 9 of [**10**]) which provides another proof that the fixed point property holds.

THEOREM 7. *If M is a locally connected 2-cell-like continuum, then M can be embedded in the plane.*

PROOF. In order to show that M can be embedded in the plane it is sufficient to show that M contains neither a homeomorphic image of V or a homeomorphic image of W (Theorem 3.6 of [**9**]). Several observations will make the proof relatively easy. First note that the curve X used to describe V contains a simple closed curve J that has the property that if g is an embedding of X into a 2-cell, then $g(J)$ separates $g(0, 1/2)$ from $g(-2^{-1/2}, -2^{-1/2})$. The curve Y contains a simple closed curve K such that if g is an embedding of Y into a 2-cell, then $g(K)$ separates $g(0, 0)$ from $g(-1, 0)$. If C is a true cyclic element of M and A is an arc of M intersecting C in only one point p, then by Theorem 3 p is a boundary point of C.

For each positive integer n let $J(n)$ denote the simple closed curve

in $V(n)$ corresponding to J, a simple closed curve separating $A(n-1)$ from $A(n)$ in V. Suppose there was a homeomorphism g from V into M. Each set $g(V(n))$ is contained in a true cyclic element $C(n)$ of M. If $g(A(1))$ intersects the interior of the curve $g(J(1))$ in $C(1)$ then $g(V(n))$ is a subset of $C(1)$ for each positive integer n. But then there is a subarc of $g(A(0))$ which intersects $C(1)$ in only one point, an interior point, which contradicts Theorem 3. It cannot be true that $g(V(n+1))$ is exterior to $g(J(n))$ and both are subsets of $C(1)$ for every positive integer n since the diameters of the sets $g(J(n))$ form a sequence converging to 0. If $g(V(n+1))$ is interior to $g(J(n))$ for some n and both are subsets of $C(1)$, exactly the same contradiction would occur as was obtained when $A(1)$ was assumed to intersect the interior of $J(1)$. Let n be the smallest integer such that $g(V(n))$ is not a subset of $C(1)$. Then $g(A(n))$ intersects the boundary of $C(n)$ and $g(V(k))$ is interior to $J(n)$ for each positive integer k greater than n, which is impossible.

A similar argument can be given to show that M contains no continuum homeomorphic to W.

This completes the proof that M can be embedded in the plane.

The referee has pointed out that Theorems 5 and 7 show that each locally connected 2-cell-like continuum is a base set. One of the formulations of the definition of a base set given by Whyburn [**10**, page 172] is that a base set is a continuum homeomorphic to a plane locally connected continuum each of whose true cyclic elements is a 2-cell.

3. **Locally connected 2-sphere-like continua.** The methods used to describe the true cyclic elements of locally connected 2-cell-like continua will be used to characterize the locally connected 2-sphere-like continua.

THEOREM 8. *A 2-sphere cannot be a proper subset of a 2-sphere-like continuum.*

PROOF. Fort and Segal [**5**] have shown that every locally connected 2-dimensional 2-sphere-like continuum is a 2-sphere. Their proof can be adapted to this theorem. If a 2-sphere S is a proper subset of a 2-sphere-like continuum M, then there is a positive number ϵ such that if f is an ϵ-map from M into a 2-sphere, $f(S)$ is a proper subset of $f(M)$. But this contradicts the Borsuk-Ulam Theorem since there is then a map of a 2-sphere into the plane under which no pair of antipodal points has the same image.

THEOREM 9. *Each true cyclic element C of a locally connected 2-sphere-*

like continuum is either a 2-sphere or a 2-cell.

PROOF. Necessarily C is locally connected, cyclic, unicoherent and contains no primitive skew curve. Then C can be embedded in a 2-sphere and is either a 2-sphere or a 2-cell, by Theorem 4.

THEOREM 10. *No true cyclic element of a 2-sphere-like locally connected continuum is a 2-cell.*

PROOF. Suppose that the true cyclic element C of the locally connected 2-sphere-like continuum M is a 2-cell. The interior of C is an open subset of M. The boundary of C is the union of four arcs $A(1)$, $A(2)$, $A(3)$ and $A(4)$ having at most endpoints in common, such that the pairs $A(1)$ and $A(3)$, and $A(2)$ and $A(4)$ are disjoint. Each component of the complement of C has exactly one limit point in C. There are closed subsets $X'(1)$, $X'(2)$, $X'(3)$ and $X'(4)$ of M whose union is the union of the boundary of C and $M-C$, such that $X'(n)$ contains $A(n)$, $n=1, 2, 3, 4$ and the pairs $X'(1)$ and $X'(3)$, and $X'(2)$ and $X'(4)$ are disjoint. There is a positive number δ such that the closures of the δ-neighborhoods of $X'(1)$ and $X'(3)$ and the δ-neighborhoods of $X'(2)$ and $X'(4)$ do not intersect. There is a subset C' of C which is a 2-cell, does not intersect the boundary of C, and contains $C-S(\text{Bdry } C, \delta)$. There is a positive number ϵ less than δ and less than the distance from the boundary of C to C'. If f is an ϵ-map from M into a 2-sphere, then $f(\text{Bdry } C)$ contains a simple closed curve which does not intersect the connected set $f(C')$ and to which the set $f(M-C')$ can be retracted. Therefore $f(M)$ is not a 2-sphere, and M is not 2-sphere-like.

Theorems 9 and 10 imply that every 2-sphere-like locally connected continuum is either a 2-sphere or a dendrite. A consequence of this is that a locally connected 2-sphere-like continuum is a 2-sphere if cyclic or 2-dimensional [**7**]. Such a continuum is also a cactoid since all its true cyclic elements are 2-spheres. (In [**10**] a locally connected continuum is defined to be a cactoid if each of its true cyclic elements is a 2-sphere.) A construction will be outlined to show that every nondegenerate dendrite is 2-sphere-like. It is easy to show that if M is a dendrite and ϵ is a positive number, there is a retraction r of M which is an ϵ-map such that $r(M)$ is homeomorphic to a polyhedral dendrite [**1**, Lemma 5.3]. Therefore it will be sufficient to show that each nondegenerate polyhedral dendrite is 2-sphere-like.

THEOREM 11. *If M is a nondegenerate polyhedral dendrite, ϵ is a positive number and p is an endpoint of M, then there is an ϵ-map from*

M onto a 2-cell such that $f(p)$ is on the boundary of $f(M)$ and the boundary of $f(M)$ is a subset of the image of $S(p, \epsilon)$.

PROOF. Proof will be by induction on the number of points of order greater than 2. If there are no points of order greater than 2, M is an arc and the theorem is clearly true. Suppose the theorem is true for all polyhedral dendrites with n or less points of order more than 2. Let M be a polyhedral dendrite with exactly $n+1$ points of order more than 2, ϵ a positive number and p an endpoint of M. Denote as q the point of order more than 2 of M not separated from p by any other point of M of order greater than 2. Let k denote the order of q. There is a k-od Q contained in M whose branch point is q such that Q contains no end point of M and no point of order more than 2 other than q. There is an $(\epsilon/2)$-map f_0 of Q onto a 2-cell with $k-1$ holes such that $f_0(a)$ is on the boundary of $f_0(Q)$ for each endpoint a of Q and each boundary component of $f_0(Q)$ is a subset of $f_0(S(a, \epsilon/2))$ for some endpoint a of Q. The closure of each of the k components of $M-Q$ is a polyhedral dendrite with not more than n points of order greater than 2. One can use the inductive hypothesis to "fill up the holes" and map the closure of the component of $M-Q$ containing p onto an annulus. That is, there is an ϵ-map f from M onto a 2-cell as asserted in the statement of the theorem.

THEOREM 12. *Suppose M is a nondegenerate polyhedral dendrite and S is a 2-manifold (with or without boundary). Then M is S-like.*

PROOF. For each positive number ϵ there is an $(\epsilon/2)$-map f of $(M-A)$ onto a sub-2-cell of S where A is an arc of diameter less than $\epsilon/2$ of M containing exactly one endpoint of M and no branch point. The map f can be chosen so that f can be extended to an ϵ-map of M onto all of S.

COROLLARY 13. *Suppose M is a nondegenerate dendrite and S is a 2-manifold (with or without boundary). Then M is S-like.*

THEOREM 14. *A locally connected nondegenerate continuum is 2-sphere-like if and only if a dendrite or a 2-sphere.*

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