$T^*W^*$. If, then, $TX \neq W$, there is a sequence $w_n^*$ in $W^*$ such that $\|w_n^*\| \to 1$, while $T^*w_n^* \to 0$. $T^*$ being 1-1 on $W^*$, it cannot be an open mapping onto $T^*W^*$, whence the last subspace is not closed.

References


University of Illinois

A SHORT PROOF OF JACOBI'S FOUR SQUARE THEOREM

L. CARLITZ

Let $R_4(n)$ denote the number of representations of $n$ as a sum of four squares and let $R'_4(4m)$, where $m$ is odd, denote the number of representations of $4m$ as a sum of four odd squares. It is familiar that

(1) $R'_4(4m) = 16\sigma(m)$

and

(2) $R_4(n) = \begin{cases} 8\sigma'(n) & \text{(n odd)}, \\ 24\sigma'(n) & \text{(n even)}, \end{cases}$

where

$$\sigma(n) = \sum_{d|n} d, \quad \sigma'(n) = \sum_{d|n; \text{d odd}} d.$$ 

These results can be proved rapidly as follows. In the usual notation of elliptic functions put [2, Chapter 21]

$$\lambda = k^2 = \frac{\theta_2^4}{\theta_3^4}, \quad 1 - \lambda = \frac{\theta_0^4}{\theta_4^4}.$$ 

Then

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Now, it follows easily from $q = \exp[-\pi K'/K]$ that [2, p. 521]
\[
\frac{1}{q} \frac{dq}{d\lambda} = -\frac{\pi}{K^2} \left( \frac{dK'}{d\lambda} - K' \frac{dK}{d\lambda} \right)
\]
\[
= \frac{1}{\lambda(1 - \lambda) \theta_3^4}.
\]
Thus logarithmic differentiation of (3) yields
\[
\theta_2^4 = \lambda \theta_3^4 = 16 \sum_{1}^{\infty} \frac{(2n - 1)q^{2n-1}}{1 - q^{2(2n-1)}} = 16 \sum_{m=1; m \text{ odd}}^{\infty} \sigma(m) q^m
\]
and (1) follows at once.
Similarly from
\[
\lambda = 2q \prod_{1}^{\infty} \left( \frac{1 + q^{2n}}{1 + q^{2n-1}} \right)^8
\]
we get
\[
\theta_0^4 = (1 - \lambda) \theta_3^4 = 1 + 8 \sum_{1}^{\infty} \left( \frac{2nq^{2n}}{1 + q^{2n}} - \frac{(2n - 1)q^{2n-1}}{1 + q^{2n-1}} \right).
\]
Replacing $q$ by $-q$ this becomes
\[
\theta_3^4 = 1 + 8 \sum_{1}^{\infty} \left( \frac{2nq^{2n}}{1 + q^{2n}} + \frac{(2n - 1)q^{2n-1}}{1 - q^{2n-1}} \right)
\]
\[
= 1 + 16 \sum_{n=1}^{\infty} \sigma'(n) q^{2n} + 8 \sum_{n=1}^{\infty} \sigma'(n) q(n)
\]
and (2) follows at once.
For the standard elliptic function proof of (2) see for example [1, pp. 205–206].

REFERENCES


DUKE UNIVERSITY