A power product $P = y_{i(1)} y_{i(2)} \cdots y_{i(n)}$, where $y_{i(j)}$ is the $i(j)$th derivative of $y$, has weight $w = \sum_{j=1}^{n} i(j)$ and degree $d = n$. The sequence of integers $(e_1, e_2, \cdots, e_n)$, where $e_k = \sum_{j=1}^{k} i(k) - k(k - 1)$, is called the weight sequence of $P$. According to a result of H. Levi ([1], Theorem 1.2, p. 545), if some $e_k < 0$, then $P \equiv 0[y^2]$. The discovery by D. G. Mead [2] that $Q = y_1 y_3 y_4 y_5 \equiv 0[y^2]$ shows that this criterion is not a necessary condition for membership in $[y^2]$. $Q$ is an unusual power product; that is, $Q$ is in $[y^2]$ and has a nonnegative weight sequence. By [3], there also exist unusual power products for $[y^p]$, $p > 2$; however, this note is concerned only with $[y^2]$. All unusual power products with weight sequences consisting of 0's, 1's, and 2's are described by Theorem IV, [2]:

A power product $P$ with weight sequence $(e_1, \cdots, e_n)$, $0 \leq e_i \leq 2$, $i = 1, \cdots, n$, is unusual if and only if somewhere in the sequence at least one of the following patterns appears: $1, 2, 2, 1; 1, 2, 2, 2, 2, 0; 0, 2, 2, 2, 1; 0, 2, 2, 2, 2, 2, 0$.

For particular arrangements of 0's, 1's, 2's, and 3's, similar results may be obtained; for example, using the notation of [2],

**Theorem A.** Let $g(a) = m(1, 3_1, 3_2, \cdots, 3_a, 2, 0)$, $[3_i = 3]$, then $g(a) = 0$ if and only if $a = 3$.

The purpose of this note is to show that, in general, a finite list of patterns will not suffice to describe unusual power products. By Theorem B there are unusual power products of arbitrarily high degree with no proper factors in $[y^2]$. The following results from [2] are stated for easy reference:

(a) $m(A, 0; B, 0) = m(A, 0)m(B, 0)$ for any sequences $A$ and $B$
(b) $m(1, 0) = -2$
(c) $m(1, 1, C) = -m(1, C)$ for any sequence $C$
(d) $m(1, 2, 1, D) = 2m(1, D)$ for any sequence $D$
(e) $m(0, E) = m(E)$ for any sequence $E$
(f) $m(1, 2, 2, F) = 2m(2, F) + m(1, 2, F)$ for any sequence $F$.

**Theorem B.** Let $f(a) = m(1, 2, 3_1, 3_2, \cdots, 3_a, 1)$, $[3_i = 3]$; then $f(a) = 0$ for all $a \geq 1$. Furthermore, if $Q$ has the weight sequence $(1, 2, 3_1, \cdots, 3_a, 1)$, $a \geq 5$, then $Q$ has no proper factor in $[y^2]$.

Received by the editors May 17, 1965.

1 This paper was written while the author held a fellowship from the American Association of University Women.
Proof. The theorem rests on two easy lemmas.

Lemma I. \( m(1, 2, 3, 3, S) = -m(1, 2, 3, S) \), for any sequence \( S \).

Proof. \( (1, 2, 3, 3, S) \) is the weight sequence of \( Q = y_1 y_3 y_8 y_6 \cdots \). Replacing \( y_6 y_8 \) and using (a)-(d), (f), we get the equation

\[
m(Q) = -2m(3, S) - 2m(1, 3, S) - 2m(2, 3, S) - m(1, 2, 3, S).
\]

Use the equation \( m(2, 3, S) = -m(3, S) - m(1, 3, S) \) to complete the proof.

In a similar fashion we prove

Lemma II. \( m(1, 2, 3, 2, S) = 2m(1, 2, S) \), for any sequence \( S \).

Returning to the proof of Theorem B, by Lemma I, \( f(a) = (-1)^{e-1}m(1, 2, 3, 1) \). But \( (1, 2, 3, 1) \) represents the same power product as \( (1, 2, 2, 1) \), namely \( R = y_1 y_2 y_4 y_6 \); and by Theorem IV of [2], \( R \equiv 0[y^2] \).

If \( P \) is a proper factor of \( Q \), then \( P \) is either an \( \alpha \)-term or a factor of

\[
\begin{align*}
P_1(k) &= y_1 y_3 y_4 y_6 y_8 \cdots y_{2k}, \\
P_2(k) &= y_1 y_3 y_4 y_6 y_8 \cdots y_{2k}, \\
P_3(k) &= y_1 y_3 y_4 y_8 y_{10} \cdots y_{2k}, \\
P_4(k) &= y_0 y_3 y_5 y_6 y_8 \cdots y_{2k}, \\
P_5(k) &= y_1 y_2 y_6 y_8 \cdots y_{2k}.
\end{align*}
\]

\( P_1(k) \) is comparable to the \( \alpha \)-term \( y_1 y_3 y_6 \) by Lemma I. \( P_2(k) \) has the weight sequence \((1, 2, 2, \cdots, 2, 0)\); hence, is in \([y^2]\) if and only if \( k = 4 \). By Lemma II, \( m(P_3(k)) = 2m(P_5(k-2)) \); and by (e) and (c) respectively, \( m(P_4(k)) = m(P_2(k-1)) \) and \( m(P_5(k)) = -m(P_3(k-1)) \). Thus \( Q \) has no proper factor in \([y^2]\).

Bibliography


University of Washington