ON POLYNOMIALS CHARACTERIZED BY A CERTAIN MEAN VALUE PROPERTY

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Let \( V \) denote the vector space of continuous real valued functions \( f(x) \) satisfying the mean value property

\[
\frac{1}{N} \sum_{i=1}^{N} f(x + ty_i)
\]

for \( x \in \mathbb{R} \), \( 0 < t < \varepsilon_x \) (\( \mathbb{R} \) denotes an \( n \)-dimensional region; \( x \) and \( y_i \) are abbreviations for \((x_1, \ldots, x_n), (y_{i1}, \ldots, y_{in})\)). We assume that the \( y_i \)'s span \( E_n \) so that \( 1 \leq n \leq N \). We furthermore assume, without loss of generality, that \( y_1, \ldots, y_n \) are linearly independent.

Friedman and Littman [5] have recently shown that \( V \) consists of polynomials of degrees \( \leq N(N-1)/2 \). This bound is actually attained when the \( y_i \)'s form the \( N \) vertices of an \((N-1)\)-dimensional regular simplex [see 4, p. 264]. On the other hand it is known that for \( n = 2 \), \( \deg f \leq N \) [see 4, Theorem 3.2]. The object of this paper is to obtain bounds on \( \deg V \) and \( \dim V \), the bounds depending on \( N \) and \( n \) \((1 \leq n \leq N)\). We use the term \( \deg V \), to denote the maximum degree of the polynomials contained in \( V \). We also characterize for fixed \( N \) and varying \( n \) \((1 \leq n \leq N)\) those configurations for which \( \deg V \) and \( \dim V \) attain their maximum.

**Theorem.** We have

\[
\deg V \leq \sum_{j=1}^{n} (N - j), \quad \dim V \leq \prod_{j=0}^{n-1} (N - j)
\]

so that for fixed \( N \) and varying \( n \) \((1 \leq n \leq N)\)

\[
\deg V \leq \frac{N(N-1)}{2}, \quad \dim V \leq N!.
\]

The latter bounds are obtained if and only if

\[
n = N \quad \text{or} \quad n = N - 1 \quad \text{and} \quad \sum_{i=1}^{N} y_i = 0.
\]

**Remark.** The bounds in (2) are not best possible. For instance, we have stated above that for \( n = 2 \), \( \deg V \leq N \) and this bound is best
possible. For fixed $n$ and $N$ the problem of determining the maximum values of $\deg \vee$, $\dim \vee$ and the configurations for which these maximum values are attained remains open.

**Proof.** We employ the following notation.

$$\frac{\partial}{\partial x} = \left( \frac{\partial}{\partial x_1}, \cdots, \frac{\partial}{\partial x_n} \right), \quad x \cdot y = x_1y_1 + \cdots + x_ny_n,$$

$$P_k(x) = \sum_{i=1}^{N} (x \cdot y_i)^k \quad (1 \leq k < \infty).$$

It is shown in [5] that (1) is equivalent to the infinite system of homogeneous partial differential equations

$$(4) \quad P_k \left( \frac{\partial}{\partial x} \right) f = 0 \quad (1 \leq k < \infty)$$

and that $\vee$, which is thus the solution space of (4), is a finite dimensional space consisting of polynomials. Let $R$ denote the ring of polynomials in $x_1, \cdots, x_n$ with real coefficients and let $\mathfrak{B}$ denote the ideal generated by the $P_k$'s $(1 \leq k < \infty)$. $R$, $\mathfrak{B}$, and $\vee$ are vector spaces over the reals and it is known that $R$ is the direct sum of $\mathfrak{B}$ and $\vee$, i.e. $R = \mathfrak{B} \oplus \vee$ [see 2, p. 53]. Thus the vector spaces $R/\mathfrak{B}$ and $\vee$ are isomorphic $(R/\mathfrak{B} \cong \vee)$.

$\deg \vee$ and $\dim \vee$ will thus be determined if we know all the polynomials in $\mathfrak{B}$. We introduce the new variables $\xi_i = x \cdot y_i$ $(1 \leq i \leq N)$. Since the $y_i$'s $(1 \leq i \leq N)$ are linearly independent we must have

$$\xi_{n+k} = \sum_{i=1}^{n} a_{ki} \xi_i \quad (1 \leq k \leq N-n)$$

for an appropriate choice of real $a_{ki}$'s. Let $R'$ denote the ring of polynomials in $\xi_1, \cdots, \xi_n$ with real coefficients and let $\mathfrak{B}'$ denote the ideal generated by the $\eta_k$'s where $\eta_k = \sum_{i=1}^{n} \xi_i^k \quad (1 \leq k \leq \infty)$. We adopt the following notation:

$$\xi = (\xi_1, \cdots, \xi_n), \quad i = (i_1, \cdots, i_n), \quad \xi_i = \xi_1^{i_1} \cdots \xi_n^{i_n}.$$

It is known [see 1, p. 41] that every polynomial $Q(\xi)$ can be expressed as

$$(5) \quad Q(\xi) = \sum' R_i \xi_i,$$

where the summation in $\sum'$ extends over those $i$'s for which $0 \leq i_j \leq N-j \quad (1 \leq j \leq n)$ and $R_i$ is a polynomial in $\eta_1, \cdots, \eta_N$. This representation is unique for $n = N$. Let $c_i$ denote the constant term in $R_i$ and let $S_i = R_i - c_i$. Clearly $S_i \in \mathfrak{B}'$. It follows from (5) that $Q(\xi) = \sum' C_i \xi_i + \sum' S_i \xi_i$ so that
\[ Q(\xi) = \sum' c_i \xi^i \pmod{\Psi'}. \]

As there are \( \prod_{j=0}^{N-j-1} (N-j) \) distinct \( \xi' \)'s, (6) shows that \( \dim R'/\Psi' \leq \prod_{j=0}^{N-j-1} (N-j) \). Since \( \sqcap \cong R/\Psi \cong R'/\Psi' \) we have \( \dim \sqcap \leq \prod_{j=0}^{N-j-1} (N-j) \). It follows furthermore from (6) that if \( Q \) is homogeneous and \( \deg Q > \sum_{j=1}^{n} (N-j) \), then \( Q \in \Psi' \). This implies that if \( P(x) \) is homogeneous and \( \deg P > \sum_{j=1}^{n} (N-j) \) then \( P \in \Psi \). Thus \( \deg \sqcap \leq \sum_{j=1}^{n} (N-j) \).

If \( n \leq N-2 \), then we conclude from (2) that \( \deg \sqcap < N(N-1)/2 \), \( \dim \sqcap < N! \). It remains to treat the two cases: (a) \( n = N \), (b) \( n = N-1 \). In case (a) the \( \xi' \)'s form a basis for \( R'/\Psi' \). For suppose that \( \sum' c_i \xi^i = 0 \pmod{\Psi'} \) for some choice of real \( c_i \)'s. Then \( \sum' c_i \xi^i = \sum_{j=1}^{n} T_j(\xi) \eta_j \) where the \( T_j \)'s are polynomials in \( \xi_1, \cdots, \xi_n \). But each \( T_j \) has a representation \( (5) \). I.e. \( T_j(\xi) = \sum' R_j(\eta) \xi^i \) where the \( R_j \)'s are polynomials in \( \eta_1, \cdots, \eta_n \). Thus \( \sum' c_i \xi^i = \sum_{j=1}^{n} \sum' R_j(\eta) \xi^i = \sum' (\sum_{j=1}^{n} R_j(\eta_j)) \xi^i \).

Since the representation \( (5) \) is unique for \( n = N \) we have

\[ c_i = \sum_{j=1}^{n} R_{j,i} \eta_j. \]

The left side of (7) is void of \( \eta \)'s so that all \( R_{j,i} \)'s and \( c_i \)'s equal 0. Thus \( \dim R'/\Psi' = N! \) and since \( \sqcap \cong R/\Psi \cong R'/\Psi' \), \( \dim \sqcap = N! \). Now \( \prod_{j=1}^{N-j-1} (N-j) \) has degree \( N(N-1)/2 \) and \( \in \Psi' \). This implies that there exists a homogeneous polynomial of degree \( N(N-1)/2 \in \Psi' \). Hence \( \deg \sqcap = N(N-1)/2 \).

If \( n = N-1 \) then we distinguish two cases. If \( \sum_{i=1}^{N} y_i \not= 0 \), then it follows from [1, Theorem 2.2] that there exists an orthogonal transformation \( x = Tx' \) such that \( g(x') = f(Tx') \) is independent of \( x_n \) and satisfies the equation

\[ g(x') = \frac{1}{N} \sum_{i=1}^{N} g(x_{p,i} + ty_{p,i}), \]

where \( y_i = Ty_i', \ x_{p,i} = (x_1', \cdots, x_{i-1}', x_{i+1}', \cdots, x_n') \), \( y_{p,i} = (y_1', \cdots, y_{i-1}', y_{i+1}', \cdots, y_n') \). Let \( \sqcap' \) denote the solution space of (8). Clearly \( \deg \sqcap = \deg \sqcap' \), \( \dim \sqcap = \dim \sqcap' \). It follows from (2) that \( \deg \sqcap = \deg \sqcap' < N(N-1)/2 \), \( \dim \sqcap = \dim \sqcap' < N! \). If \( \sum_{i=1}^{N} y_i = 0 \), then define

\( \tilde{x} = (x_1, \cdots, x_n, x_{n+1}), \ y_i = (y_{i1}, \cdots, y_{in}, 1) (1 \leq i \leq N) \), \( F(\tilde{x}) = f(x) \).

We notice that \( \sum_{i=1}^{N} \tilde{y}_i \not= 0 \). It therefore follows from [3, Theorem 2.2] that \( \sqcap \) is identical with the solution space \( \sqcap \) of

\[ F(\tilde{x}) = \frac{1}{N} \sum_{i=1}^{N} F(\tilde{x} + t\tilde{y}_i). \]
Equation (9) is included in Case (a). It follows that \( \text{deg} \, \nabla = \text{deg} \, \nabla = N(N-1)/2 \), \( \text{dim} \, \nabla = \text{dim} \, \nabla = N! \).

**References**


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