ELEMENTARY PROOF OF A THEOREM ON CONFORMAL RIGIDITY

EDGAR REICH

Let \( f(z) \) be single valued and analytic in the annulus \( A: a < |z| < b \), \( f(A) \subset A \), and suppose that \( f \) maps every closed curve with winding number +1 about \( z = 0 \) onto a curve with the same property. Then \([1], [2], [3], [4]\) \( f \) if a rotation. Stimulated by a discussion with Professor A. Marden we shall give a short proof of this result with a minimum of topological notions.

By hypothesis, the integral of

\[
\frac{1}{z} - \frac{f'(z)}{f(z)}
\]

around any closed curve in \( A \) vanishes. Hence there exists a branch, \( F(z) \), of \( \log [z/f(z)] \) single valued in \( A \). Let us define

\[
\begin{align*}
u(re^{i\theta}) &= \text{Re } F(re^{i\theta}) = \log r - \log |f(re^{i\theta})|, \\
I(r) &= (2\pi)^{-1} \int_{0}^{2\pi} \nu(re^{i\theta}) \, d\theta.
\end{align*}
\]

By Cauchy's theorem applied to \( z^{-1}F(z) \), \( I(r) \) is independent of \( r \). On the other hand, since \( f(A) \subset A \),

\[
\log(r/b) \leq \nu(re^{i\theta}) \leq \log(r/a), \quad a < r < b.
\]

Therefore

\[
\begin{align*}
\limsup_{r \to a} I(r) &\leq 0, \\
\liminf_{r \to b} I(r) &\geq 0.
\end{align*}
\]

Thus, \( I(r) \equiv 0, a < r < b \). Hence, by (1),

\[
\begin{align*}
J(r) &= (2\pi)^{-1} \int_{0}^{2\pi} \nu(re^{i\theta}) |d\theta = (2\pi)^{-1} \left[ \int_{u \geq 0} u \, d\theta - \int_{u \leq 0} u \, d\theta \right] \\
&= -2(2\pi)^{-1} \int_{u < 0} u \, d\theta \leq 2 \log(b/r),
\end{align*}
\]

or alternatively,

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(3) \[ J(r) = 2(2\pi)^{-1} \int_{u>0} u \, d\theta \leq 2 \log(r/a), \quad a < r < b. \]

Since \(|u(re^{i\theta})|\) is subharmonic, \(J(r)\) is a convex function of \(\log r\). Therefore \(J(r) \leq \max\{J(r_1), J(r_2)\}\) whenever \(a < r_1 \leq r \leq r_2 < b\). Letting \(r_1 \to a\), \(r_2 \to b\), and using (2), (3) we obtain
\[ J(r) = 0, \quad a < r < b. \]
Thus \(u \equiv 0\), \(|z^{-1}f(z)| \equiv 1\). Thus \(f(z) = e^{ic}z\), \(c\) real.

References


University of Minnesota