IDEALS GENERATED BY PRODUCTS

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In [1], Levi obtained results on the structure of the differential ideals \([y^n]\) and \([uv]\) and applied these results to the component theory of differential polynomials. The present paper uses Levi’s methods to extend his main results on \([uv]\) to \([y_1y_2 \cdots y_n]\). Since the \(y_i\) are independent indeterminates, \([y_1 \cdots y_n]\) is related to \([y^n]\) but is not quite a generalization. The results are motivated by and apply to differential algebra; however, we follow Levi’s suggestion at the close of [1] in stating them in the more general form in which \(y_{ij}\) is not necessarily a derivative of \(y_i\).

Let \(y_{ij}\) \((i=1, \cdots, n\) and \(j=0, 1, \cdots)\) be a set of independent indeterminates over a field \(F\) and let \(R\) be the polynomial ring in the \(y_{ij}\) over \(F\). The signature of a monomial \(M\) in \(R\) is \(D=(d_1, \cdots, d_n)\) if \(M\) has degree \(d_i\) in the \(y_{ij}\) with \(i=h\). The weight of \(M\) is the sum of the \(j\)'s for the factors \(y_{ij}\) of \(M\). A polynomial of \(R\) is homogeneous with signature \(D\) if each of its terms has this signature; it is isobaric of weight \(w\) if each term has weight \(w\).

For \(j=0, 1, \cdots\) let \(x_j\) be a linear combination, with nonzero coefficients in \(F\), of all the products \(y_{i_1j_1}y_{i_2j_2} \cdots y_{i_nj_n}\) of weight \(j\). Let \(I_t\) be the ideal \((x_0, x_1, \cdots, x_t)\) in \(R\). Let \(I=(x_0, x_1, \cdots)\) and let \(Q\) be the quotient ring of \(R\) modulo \(I\). Below we describe functions \(f(D)\) and \(g(D)\) such that a monomial with signature \(D\) and weight \(w\) is in \(I_t\) if \(w<f(D)\) and \(t\geq g(D)\) and such that for every \(D\) and \(w\geq f(D)\) there is a monomial with signature \(D\) and weight \(w\) that is not in \(I\).

2. Levi bases. In the case \(n=2\), Levi obtained the following bases for \(Q\) and \(R\) as vector spaces over \(F\). Let \(u_j=y_{1j}\) and \(v_j=y_{2j}\). A product
\[
P = u_{i_1} \cdots u_{i_r} v_{j_1} \cdots v_{j_s}
\]
(1)
of signature \((r, s)\) is an \(\alpha\) term if \(s=0\) or \(j_i \geq r\) and a \(\beta\) term otherwise. An \(\lambda\) term is of the form \(AX\) with \(A\) an \(\alpha\) term and \(X\) a power product in the \(x_j\). \((X\) may be 1; thus the \(\lambda\) terms include Levi’s \(\alpha\) and \(\gamma\) terms.) Levi showed that the \(\alpha\) terms are a basis for \(Q\) and the \(\lambda\) terms for \(R\).

Let \(P'\) be a factor of \(P\) in (1) selected as follows. If \(P\) is an \(\alpha\) term,
\( P' = P \). If \( P \) is a \( \beta \) term, let \( e = j_i + 1 \), \( S = u_i v_i \), and \( P' = P / S \). Let \( P_i \) be defined by \( P_0 = P \) and \( P_{i+1} = P'_i \). Let \( \theta(P) \) be the expression for \( P \) in the form \( A S_1 \cdots S_a \) where \( a \) is the first \( i \) for which \( P_i \) is an \( \alpha \) term, \( A = P_a \), and \( S_i = P_i / P_{i-1} \). The weight \( h_i \) of \( S_i \) satisfies \( h_1 \leq h_2 \leq \cdots \leq h_a \). Let \( \lambda(P) = Ax_{h_1} \cdots x_{h_a} \). Levi showed that \( P \to \lambda(P) \) is a 1-1 mapping of the set of all power products in the \( u_j \) and \( v_j \) onto the set of \( \lambda \) terms.

3. Extension to general \( n \). Let \( P = Y_1 Y_2 \cdots Y_n \) where \( Y_k \) is a power product in the \( y_{ij} \) with \( i = k \). For \( 2 \leq k \leq n \) we inductively define \( \alpha_k \) terms in the \( y_{ij} \), \( \cdots \), \( y_{kj} \) and an expression \( \theta_k \) for \( Y_1 \cdots Y_k \). When \( k = 2 \), \( \alpha_2 \) terms and \( \theta_2 = \theta(Y_1 Y_2) \) are defined as in the previous section with \( y_{1j} \) in the role of \( u_j \) and \( y_{2j} \) in that of \( v_j \). We assume the definition of \( \alpha_m \) terms and of \( Y_1 \cdots Y_m \) in the form

\[
\theta_m = A_m S_{m1} \cdots S_{mb}
\]

where \( A_m \) is an \( \alpha_m \) term and \( S_{mj} \) has signature \((1, \cdots, 1, 0, \cdots, 0)\), with \( m \) ones, and weight \( h_{mj} \) satisfying \( h_{m1} \leq h_{m2} \leq \cdots \leq h_{mb} \). Thinking of \( S_{mj} \) as \( u_j \) and \( y_{m+1,j} \) as \( v_j \), let \( \theta(S_{m1} \cdots S_{mb} Y_{m+1}) = A^* S_{m+1,1} \cdots S_{m+1,c} \). We then define \( \theta_{m+1} \) to be

\[
A_{m+1} S_{m+1,1} \cdots S_{m+1,c}
\]

where \( A_{m+1} = A_m A^* \). The power product \( Y_1 \cdots Y_{m+1} \) is defined to be an \( \alpha_{m+1} \) term if \( c \) in (2) is zero, i.e., if \( Y_1 \cdots Y_{m+1} = A_{m+1} \).

As before a \( \lambda \) term is of the form \( AX \) with \( A \) an \( \alpha \) (i.e., \( \alpha_n \)) term and \( X \) a power product in the \( x_j \). Let \( \theta(P) = \theta_n = A S_1 \cdots S_c \) with \( A \) an \( \alpha \) term and \( S_j \) of weight \( h_j \) satisfying \( h_1 \leq \cdots \leq h_c \) and let \( \lambda(P) = Ax_{h_1} \cdots x_{h_c} \). The following outline of the process of showing that the \( \alpha \) terms are a basis for \( Q \) and the \( \lambda \) terms for \( R \) is essentially as in Levi's work:

Since \( x_{h_1} \) is a linear combination of power products one of which is \( S_1 \), \((P/S_1)x_{h_1} \equiv 0 \) (mod 1) can be solved as

\[
P = f_1 N_1 + \cdots + f_s N_s \pmod{1}
\]

where the \( f_i \) are in \( F \) and the \( N_i \) are power products in the \( y_{ij} \). Any \( N \) in (3) which is not an \( \alpha \) term can be replaced by an expression (3) in which \( N \) plays the role of \( P \). Continuing the process, one ends in a finite number of steps with \( P \) congruent to a linear combination of \( \alpha \) terms of the same signature and weight as \( P \). Thus the \( \alpha \) terms generate \( Q \). It follows from the process of obtaining the \( \theta_k \) and the 1-1 character of \( P \to \lambda(P) \) in the case \( n = 2 \) that, in the case of general \( n \), \( P \to \lambda(P) \) is a 1-1 mapping of the set of power products in the \( y_{ij} \) onto the set of \( \lambda \) terms. This establishes that the \( \lambda \) terms are a basis for \( R \). Then the \( \alpha \) terms are linearly independent modulo 1 and form a basis for \( Q \).
The function $f(D)$ is the minimum of the weights of monomials with signature $D$ that are not in $I$; hence $f(D)$ is the minimum of the weights of the $\alpha$ terms with signature $D$. An $\alpha$ term $A$ is of the form $A_{n-1}A^*$ where $A_{n-1}$ is an $\alpha_{n-1}$ term and $A^*$ is an $\alpha_2$ term in the $S_{n-1,i}$ and the $y_{nj}$. If $D = (d_1, \cdots, d_n)$ and $A^*$ is of degree $t$ in the $S$'s, the weight of $A$ is at least

$$f(d_1 - t, \cdots, d_{n-1} - t) + td_n$$

and is (4) if $A_{n-1}$ is an $\alpha_{n-1}$ term of minimal weight for its signature and $Y_n = (y_{ni})^{d_n}$. Hence $f(D)$ is the minimum of (4) for $0 \leq t \leq \min (d_1, \cdots, d_{n-1})$.

There are other ways of defining a basis for $Q$. Although the basis may be different, the function $f(D)$ does not depend on the basis. One such alternative shows that

$$f(D) = \min [f(d_1 - r, \cdots, d_a - r) + f(d_{a+1} - s, \cdots, d_b - s) + \cdots + f(d_{c+1} - t, \cdots, d_n - t) + f(r, s, \cdots, t)]$$

where $r, s, \cdots, t$ range over all nonnegative integers such that the arguments in (5) are nonnegative.

It will be shown in another paper that $f(D)$ may be evaluated explicitly as follows. We assume without loss of generality that $d_1 \leq d_2 \leq \cdots \leq d_n$. For $2 \leq i \leq n$ let $q_i = (d_1 + \cdots + d_i)/(i-1)$ and let $k$ be the smallest $i$ for which $q_i$ assumes its minimum. Let $q$ and $r$ be integers defined by $d_1 + \cdots + d_k = (k-1)q + r$ and $0 \leq r < k-1$. Let $c_i = q - d_i$ for $i = 1, \cdots, k$. Let $\sigma_1 = c_1 + \cdots + c_k$ and $\sigma_2 = \sum_{i<j} c_ic_j$. Then $f(D) = \sigma_2 + r\sigma_1 + [(r+1)r/2]$.

It is easily seen from Levi's process that $g(d_1, d_2)$ may be chosen as $d_1 + d_2 - 2$. This and the process described above show that $g(d_1, \cdots, d_n)$ may be chosen as

$$d_n + \min (d_1, \cdots, d_{n-1}) - 2.$$ 

Using the symmetry of the ideal $I_i$ in the subscripts $i$ of the $y_{ij}$, this may be improved to $g(D) = \min_{i \neq j} (d_i + d_j) - 2$.

References