THE DIFFERENTIAL IDEAL \([uv]\)
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1. Introduction. Let \( R\{u, v\} \) be the polynomial ring \( R[u, u_1, u_2, \cdots ; v, v_{l+1} \cdots ] \) over \( R \), a field of characteristic zero, with the derivation \( D(y) = y_{i+1} \) for \( y = u \) or \( v \).

Let \( \Omega = [uv] \) be the differential ideal generated by the form \( X = uv_1. \Omega \) has the same elements as the ideal \( (uv_1, (uv_1)_1, (uv_1)_2, \cdots ) \), where the subscripts again denote derivatives.

A power product in \( R\{u, v\} \) \( P = u_i(v)\cdot u_i(v_2) \cdot \cdots \cdot u_i(m)v_j(v_2) \cdot \cdots \cdot v_j(n) \) is of weight, \( w(P) = \sum_{k=1}^{m} i(K) + \sum_{p=1}^{n} j(P), \) and signature, \( \text{sig}(P) = (m, n) \).

The following fundamental theorem is proved in [3].

**Levi's Theorem.** If \( P \) is a power product in \( R\{u, v\} \) and \( w(P) < m \cdot n \), then \( P \) is in the ideal \([uv]\).

The purpose of this paper is to show that if \( P \) contains no proper factor which is in \([uv]\), and if \( w(P) \geq mn \), then \( P \) is not in \([uv]\).

2. Derivations and isomorphic images of \( R\{u, v\} \). Computations in \( R\{u, v\} \) are simplified by working in an isomorphic image of \( R\{u, v\} \), \( R\{\bar{u}, \bar{v}\} \). \( R\{\bar{u}, \bar{v}\} \) is the ring \( R[\bar{u}, \bar{u}_1 \cdots , \bar{v}, \bar{v}_1 \cdots ] \) with derivation \( \overline{D}(\bar{y}) = \bar{y}_{i+1} \) for \( y = u \) or \( v \). The isomorphism is established by the mapping \( h: u_i \rightarrow \bar{u}_i, v_j \rightarrow \bar{v}_j \). Thus \( \overline{D}(\bar{u}_i) \) corresponds to \( D(u)_i/(i+1) \) and \( \overline{D}(\bar{v}_j) \) to \( D(v)_j/j+1 \). For typographical convenience, the bars will be omitted; hence \( \overline{D}^n(u\bar{v}) \) is written \( (uv)_n = \sum_{j=0}^{n} u_jv_{n-j} \).

**Definition 2.1.** \( D^l \) is defined on \( R[u, u_1, \cdots , v, v_{l+1} \cdots ] \) by

1. \( D^l(u_i) = (i + 1)u_{i+1} \) for \( i \geq 0 \).
2. \( D^l(v_j) = \begin{cases} (j - l + 1)v_{j+1} & \text{for } j \geq l, \\ 0 & \text{for } j < l. \end{cases} \)
3. If \( D^l_k \) has been defined, then \( D^l_{k+1} = D^l(D^l_k) \).

**Theorem 2.2.** Let \( h \) be the (nondifferential) isomorphism of \( R = R[u, u_1, \cdots , v, v_1, \cdots ] \) onto \( R_1 = R[u, u_1, \cdots , v, v_{l+1}, \cdots ] \) determined by mapping \( u_i \) into \( u_i \) and \( v_j \) into \( v_{j+1} \). Then
\[
(2.1) \quad h(D^0(P)) = D^l(h(P)).
\]

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Proof. It suffices to show (2.1) for \( P = u_i \) and \( P = v_j \). Suppose that \( i \geq 0 \), then for \( l \geq 0 \), \( h(D^0(u_i)) = h((i + 1)u_{i+1}) = (i + 1)u_{i+1} = D^1(h(u_i)) \); and for \( j \geq l \), \( h(D^0(v_j)) = h((j + 1)v_{j+1}) = (j + 1)v_{j+1} = D^1(v_{j+1}) = D^1(h(v_j)) \).

Corollary 2.3. \( \mathfrak{a}_1 \) is closed under the operation \( D^1 \). Furthermore, the ideal \([uv_1]\), the image of \([uv]\) under the mapping \( h \), is closed under \( D^1 \).

Corollary 2.4. Let \( R\{u, v_1\} \) be the Ritt algebra \((\mathfrak{r}_1, D^1)\), then \( R\{u, v_1\} \) is isomorphic to \( R\{u, v\} \).

Let \((uv)\) be the (algebraic) subring of \( R\{u, v\} \) generated by \( uv \); that is, \((uv)\) is the set of elements of \( R\{u, v\} \) divisible by \( uv \).

Theorem 2.5. There is a module isomorphism \( g \) which maps \( uR\{u, v\}/(uv) \) onto \( R\{u, v_1\} \).

Proof. Let \( I = (uv) \). If \( a \in uR\{u, v\}/(uv) \), then for a unique \( b \) not involving \( v \), \( a = ub + I \). Define \( g \) by \( g(a) = b \). Then \( g(I) = 0 \), \( g(c + I) = c/u \), if \( c \) does not contain \( v \). Clearly if \( r \in R \), \( g(ra) = rg(a) \) and if \( a_1 \) and \( a_2 \) are elements of \( uR\{u, v\}/(uv) \), then \( g(a_1 + a_2) = g(a_1) + g(a_2) \). Furthermore, for every \( c \) in \( R\{u, v_1\} \), \( g^{-1}(c) = uc + I \) and \( g^{-1}(c) \) is an element of \( uR\{u, v\}/(uv) \).

Theorem 2.6. Under \( g \), \( u[uv]/(uv) \) in \( R\{u, v\} \) is mapped isomorphically on \([uv_1]\) in \( R\{u, v_1\} \).

Proof. If \( a \in u[uv]/(uv) \), then \( a = uc + I \), where \( c = \sum_{i=0}^{m} d(i)(uv)_i \) with \( d(i) \in R\{u, v\} \). For \( i > 0 \), \((uv)_i = (uv_1)_{i-1} + u_i v \); hence, \( uc + I = u \sum_{i=0}^{m} d(i)(uv_1)_{i-1} + I \). Thus \( g(a) = c \) and \( c \in [uv_1] \). Further, \( g^{-1}g(a) = a \). If any \( c \) is in \([uv_1] \), then \( g^{-1}(c) = uc + I \), or \( u \sum_{i=0}^{m} d(i)(uv_1)_{i-1} + I \). But then certain elements of \( I \) may be used to fill out the sums because \( ud(i)u_i v \in I \) for every \( i \). Therefore \( u \sum_{i=0}^{m} d(i)(uv_1)_i + I = u \sum_{i=0}^{m+1} d(i-1)(uv)_i + I \), and \( g \) covers all of \([uv_1]\) and is an isomorphism.

Corollary 2.7. If \( Q \equiv 0[uv_1] \), then \( u \cdot Q \equiv 0 \) \([uv]\).

Proof. Using the \( g^{-1} \) of Theorem 2.2, \([uv_1]\) is mapped onto \( u[uv]/(uv) \). Hence \( uQ \equiv 0[uv] \) because \( uQ \subseteq uQ + I = g^{-1}(Q) \).

3. The operator \( T_n \). Let \( P = u_j U V \) be a power product of signature \( \langle k, l \rangle \) and excess weight zero.

Definition 3.1. \( T_n \) operates on \( V \) and is defined by

1. For \( n = 1 \), \( T_1(V) = D^1(V) - D^0(V) \).
2. If \( T_{n-1}(V) \) has been defined, then \( T_n(V) = D^1(T_{n-1}(V)) - T_{n-1}(D^0(V)) \). (Note that \( T_n \) and \( D^1 \) do not commute.)
Theorem 3.2. Let $V = v_{j(1)}v_{j(2)} \cdots v_{j(l)}$, then for $n \leq l$, $T_n(V) = (-1)^n n! \sum v_{j(1)} \cdots v_{t(n)+1} \cdots v_{t(n)+1}$, with the summation extending over all products in which exactly $n$ v-subscripts are raised by 1. (That is, no $j(i)$, $i = 1, \ldots, l$, is raised more than 1.) If $n > l$, $T_n(V) = 0$.

Proof. The proof is by induction on $n$, keeping $l$ fixed.

For $n = 1$, 
\[ T_1(V) = D^1(V) - D^0(V) \]
\[ = \sum_{m=1}^{l} (j(m))v_{j(1)} \cdots v_{j(m)+1} \cdots v_{j(l)} \]
\[ - \sum_{m=1}^{l} (j(m) + 1)v_{j(1)} \cdots v_{j(m)+1} \cdots v_{j(l)} \]
\[ = \sum_{m=1}^{l} v_{j(1)} \cdots v_{j(m)+1} \cdots v_{j(l)}. \]

For $n > 1$, assume that the theorem holds for values less than $n$. Let $Z_n$ be the set of all functions $z$ on $\{1, 2, \ldots, l\}$ to $\{0, 1\}$ with $n$ occurrences of 1. The induction hypothesis may now be written, for $p < n$, 
\[ T_p(V) = (-1)^p p! \sum_{z \in Z_p} v_{j(1)+z(1)} \cdots v_{j(l)+z(l)}. \]

By definition $T_n(V) = D^1(T_{n-1}(V)) - T_{n-1}(D^0(V))$, and the induction hypothesis may be applied to $T_{n-1}$. Using the definition of $D^0$ and $D^1$, an expression for $T_n$ may be derived as follows.

\[ T_n(V) = D^1\left((-1)^{n-1}(n - 1)! \sum_{z \in Z_{n-1}} v_{j(1)+z(1)} \cdots v_{j(l)+z(l)}\right) \]
\[ - T_{n-1}\left(\sum_{t=1}^{l} (j(t) + 1)v_{j(1)} \cdots v_{j(t)+1} \cdots v_{j(l)}\right) \]
\[ = (-1)^{n-1}(n - 1)! \]
\[ \cdot \sum_{z \in Z_{n-1}} \left(\sum_{t=1}^{l} (j(t) + z(t))v_{j(1)+z(1)} \cdots v_{j(t)+z(t)+1} \cdots v_{j(l)+z(l)}\right) \]
\[ - \sum_{t=1}^{l} (j(t) + 1)(-1)^{n-1}(n - 1)! \]
\[ \cdot \left(\sum_{z \in Z_{n-1}} v_{j(1)+z(1)} \cdots v_{j(t)+1+z(t)} \cdots v_{j(l)+z(l)}\right). \]
These two sums are exactly comparable, the same $t$'s and $z$'s occurring in each. The sign of one term is $+$ and the other $-$; the sum of the coefficients being

$$j(t) + z(t) - (j(t) + 1).$$

The sum coefficient is then $-1$ for exactly those terms where $z(t) = 0$. It is 0 for the others. For $n \leq l$, then, the terms unify, giving for each a new $z$, an element of $Z_n$; and for $n > l$, the terms cancel. In case $n \leq l$, each element of $Z_n$ can be found in $n$ ways from as many elements of $Z_{n-1}$; hence, the new factor in the coefficient is $-n$. This concludes the proof.

The $T$-operator will now be applied to an arbitrary power product $P$ of excess weight zero. First of all, if $P$ contains any factor of negative excess weight, then $P$ is in $[uv]$. Therefore, in particular, assume that $P$ does not contain $uv$.

**Theorem 3.3.** Let $P = u_1UV$, then $P = uUT_1(V)[uv]$.

**Proof.** Since $P$ is a power product of excess weight zero, $uUV$ has negative excess weight and is zero modulo $[uv]$ by Levi's Theorem. Mapping $R$ into itself by $D^0$, $uUV = 0[uv]$ becomes

$$u_1UV + uD^0(U)V + uUD^0(V) \equiv 0 [uv].$$

Consider $Q = UV$ as a power product in $R_1$. Then $S = Uh^{-1}(V)$ in $R$ has signature $(k-1, l)$ and weight $w = kl - 1 - l < (k-1)l$; hence $S \equiv 0 [uv]$. Under $D^0$, $S \equiv 0 [uv]$ becomes

$$D^0(U)(h^{-1}(V)) + UD^0(h^{-1}(V)) \equiv 0 [uv].$$

Mapping $R$ into $R_1$, (3.2) becomes

$$D^0(U)V + UD^1(V) \equiv 0 [uv_1].$$

The derivation of $R\{u, v_1\}, D'$, may be used in $[uv]$ because using the mapping $g$ of Theorem 3.5, $g^{-1}D^1g$ maps $uR\{u, v\}/(uv)$ into itself and $u[uv]/(uv)$ into itself. Hence, by Corollary 2.7,

$$uD^0(U)V + uUD^1(V) \equiv 0 [uv].$$

Substituting (3.4) in (3.1) completes the proof.

**Lemma 3.4.** Let $P = u_jUV$ and let $h$ map $R$ onto $R_1$. If $Q = Uh^{-1}(T_{j-1}(V))$, then $Q \equiv 0 [uv]$.

**Proof.** By Theorem 3.2, $w(T_{j-1}(V^*)) = w(V) + (j-1)$ for each term $T_{j-1}(V^*)$ in $T_{j-1}(V)$. For each term $Q^*$ in $Q$, $w(Q^*) = w(P) - j$
+(j−1)−l=kl−l−1<(k−1)l; and the signature of Q* is <k−1, l>.
Hence Q*≡0[uv] by Levi’s Theorem.

**Theorem 3.5.** Let \( P = u_j UV \), then for all \( j > 0 \),

\[
P \equiv \frac{1}{j!} uUT_j(V) [uv].
\]

**Proof.** The proof is by induction on \( j \), and the case \( j = 1 \) is covered by Theorem 3.3. Assume that (3.5) holds for values less than \( j \). In \( R \), \( u_{j−1} UV \equiv 0 [uv] \) by Levi’s Theorem. Under \( D^0 \), we have

\[(3.6) j u_j UV \equiv (-u_{j−1} D^0(U) V - u_{j−1} U D^0(V)) [uv].\]

Applying the induction hypothesis to each term on the right (3.6) becomes

\[
j u_j UV \equiv \left( -\frac{1}{(j−1)!} u D^0(U) T_{j−1}(V) \right) [uv].
\]

Map \( R \) onto \( R_1 \) by \( k \) and consider \( Q = U T_{j−1}(V) \) as a power product in \( R_1 \). Then \( S = U h^{−1}(T_{j−1}(V)) \) is in \([uv]\) by Lemma 3.4. Under \( D^0 \), \( S \equiv 0[uv] \) becomes

\[(3.8) D^0(U) h^{−1}(T_{j−1}(V)) + U D^0(h^{−1}(T_{j−1}(V))) \equiv 0 [uv].\]

Mapping \( R \) onto \( R_1 \), (3.8) becomes

\[(3.9) D^0(U) T_{j−1}(V) + U D^1(T_{j−1}(V)) \equiv 0 [uv_1].\]

By Corollary 2.7, we get

\[(3.10) u D^0(U) T_{j−1}(V) + u U D^1(T_{j−1}(V)) \equiv 0 [uv].\]

Substituting (3.10) in (3.7) completes the proof.

**4. The converse of H. Levi’s Theorem for [uv].** Let \( P = u_{i(1)} u_{i(2)} \ldots \cdot u_{i(k)} v_{j(1)} v_{j(2)} \ldots v_{j(l)} \) be of signature \( <k, l> \) and weight \( w \). Assume that \( P \) has no factor of negative excess weight. By Theorem III of [4], without loss of generality, we may set \( w(P) = kl \). If a sequence of \( k \) transformations exist such that

\[
(1) \ V = v_{j(1)} \ldots v_{j(l)} \ \text{is changed to} \ v'_k,
\]

\[
(2) \ \text{in the} \ t\text{th transformation exactly} \ i(t) \ v\text{-subscripts are increased by one},
\]

\[
(3) \ U = u_{i(1)} \ldots u_{i(k)} \ \text{is changed to} \ u^k;
\]
then $P$ may be written congruent to a linear combination of $\alpha$-terms of the same weight and signature as $P$, [3]. $P$ is of excess weight zero and thus $P = cu_k v_k [uv]$. The only question concerns the coefficient $c$, which is not zero, but is $(-1)^{i_1+i_2+\cdots+i_m}$ where $m$ is the number of sequences which transform $V$ to $v_k$. Thus $c = 0$ if and only if $m = 0$, and we have proved

**Theorem 4.1.** If $P = UV$ has a nonnegative weight matrix, then $P$ is not in $[uv]$ if and only if $V$ can be transformed to $v_k$ by a sequence of $n$ steps, in the $t$th of which exactly $i(t)$ $v$-subscripts are increased by one.

It remains to characterize those $U$ and $V$ for which (4.1) exists.

At the $t$th step, suppose a power product $M$ is transformed into a power product $N$ as follows: $u_t$ in $M$ is replaced with $u$ and the lowest $t$ $v$-subscripts (assuming that $j(1) \leq j(2) \leq \cdots \leq j(t) \leq \cdots \leq j(0)$) are increased by one. Now, if $N$ contains a factor with negative excess weight, then the same is true of $M$. More generally, we prove

**Theorem 4.2.** Let $M$ be a power product of signature $\langle k, l \rangle$ containing $u_t$, $t > 0$ and $t$ $v$'s, $v_{j(1)} \cdots v_{j(t)}$, and let

$$N = M \frac{uv_{j(1)} u_{i(t)} \cdots v_{j(t)} + 1}{u v_{j(1)} \cdots v_{j(t)}}.$$

Then if $G$ is any factor on $N$ with excess weight $e(G)$, there is a factor $F$ of $M$ with excess weight $e(F) \leq e(G)$.

**Proof.** We may assume $G$ has $u$ as a factor; otherwise, by reducing the subscripts in $G$ that have been raised we get a factor $G^*$ of $M$ with $e(G^*) \leq e(G)$. Therefore $G$ is of the form $u U_1 V$, where $U_1$ is a factor of $U$; notationally, let $U_1 = U$. If $V$ involves no unchanged subscripts, then lowering the $n$ subscripts of $V$ that have been raised we get $V^*$ and a factor $UV^*$ of $M$ with $e(UV^*) = w(U) + w(V) - n - (k - 1)n = e(u U V)$. If $V$ involves all the changed subscripts, then similarly $e(u U V^*) = t + w(U) + (W(V) - t) - k \ deg V^* = e(u U V)$. If $V$ involves an unchanged subscript but not all changed ones, we can exchange an unchanged subscript for a changed one except in the case that all the changed subscripts of $N$ are $j(t) + 1$ and all the unchanged subscripts of $G$ are $j(t)$. Thus a reduction is achieved except in the case that $G$ is of the form $u U v_{j(t)} v_{j(t)} v_{j(t)}$, $p < t, q > 0$. Consider the cases (1) $k \geq j(t) + 1$ and (2) $k \leq j(t)$.

In case 1,

$$e(u U v_{j(t)} v_{j(t)} + v_{j(t)}) = e(u U v_{j(t)} v_{j(t)} + v_{j(t)}).$$
In case 2,

\[ e(u U^p v_{j(t)} + v_{j(t)}^{q-1}) \leq e(u U^p v_{j(t)} + v_{j(t)}^q). \]

In either case, a factor \( F \) of \( M \) has been found such that \( e(F) \leq e(G) \), and the proof is complete.

**Corollary 4.3.** If \( P = UV \) has a nonnegative weight matrix and excess weight zero, then \( P \not\equiv 0[uv] \).

**Proof.** By symmetry we may assume that \( V \not\equiv 0(v) \). By Theorem 4.2, there is a sequence of transformations satisfying (4.1) which transforms \( P \) into the \( \alpha \)-term \( u^k v^l \).

**Corollary 4.4.** If \( P = u^i v_j \), the smallest exponent \( q \) such that \( P^q \equiv 0[uv] \) is \( q = i + j + 1 \).

**Proof.** \( Q = (u, v)^{i+j+1} \) has negative excess weight; hence, by Levi's Theorem is in \([uv]\). On the other hand, \( S = (u, v)^{i+j} \) has a nonnegative weight matrix, excess weight zero, and is not in \([uv]\) by Corollary 4.3. This solves Ritt's exponent problem for \([uv]\), ([1], p. 177).

**Bibliography**


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