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$q$-ANALOGUES OF CAUCHY’S FORMULAS

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1. Let $q$ be a given number and let $\alpha$ be real or complex. The $\alpha$th “basic number” is defined by means of $[\alpha] = (1-q^\alpha)/(1-q)$. This has served as a basis for an extensive amount of literature in mathematics under such titles as Heine, basic, or $q$-series and functions. The basic numbers also occur naturally in many theta identities. The works of Jackson (for bibliography see [2]) and Hahn [3] have stimulated much interest in this field.

One important operation that is intimately connected with basic series as well as with difference and other functional equations is the $q$-derivative of a function $f$. This is defined by

\[
Df(x) = \frac{f(qx) - f(x)}{x(q - 1)}.
\]

Jackson defined the operations, which he called $q$-integration,

\[
\int_0^x f(t)d(q, t) = x(1 - q) \sum_{k=0}^{\infty} q^k f(xq^k)
\]

and

\[
\int_x^\infty f(t)d(q, t) = x(1 - q) \sum_{k=0}^{\infty} q^{-k} f(xq^{-k})
\]

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provided the series on the right hand side are convergent. Both (1.2) and (1.3) are inverse operations to (1.1) and are analogues of the definite integrals \( \int_0^x f(t)\,dt \) and \( \int_x^a f(t)\,dt \) respectively. In fact each approach the respective integral as \( q \to 1 \) when \( f \) is Riemann integrable in the intervals of integration.

The definite \( q \)-integral \( \int_a^x f(t)\,d(q, t) \) is defined by means of

\[
\int_a^x f(t)\,d(q, t) = \int_0^x f(t)\,d(q, t) - \int_0^a f(t)\,d(q, t).
\]

The purpose of this note is to obtain \( q \)-analogues of Cauchy's formulas for multiple integrals

\[
\int_a^x \int_a^{x_n-1} \cdots \int_a^{x_1} f(t)\,dt\,dx_1 \cdots dx_{n-1}
\]

(1.5)

\[
= \frac{1}{(n-1)!} \int_a^x (x - t)^{n-1} f(t)\,dt.
\]

and

\[
\int_x^a \int_x^{x_n-1} \cdots \int_x^{x_1} f(t)\,dt\,dx_1 \cdots dx_{n-1}
\]

(1.6)

\[
= \frac{1}{(n-1)!} \int_x^a (t - x)^{n-1} f(t)\,dt.
\]

The \( q \)-analogues of (1.4) and (1.5) are given in Theorems 1 and 2 below. We remark that (3.1) and (3.2) can be regarded as a transformations of \( n \)-fold infinite series to a single infinite series. These are basically different from a finite analogue of (1.5) that has been recently given by Traub [4]. We further note that (3.1) and (3.2) can be used to define fractional \( q \)-integral and \( q \)-derivative in the same way that (1.4) and (1.5) have been used to define fractional integrals and derivative. This we shall give elsewhere.

2. Let for a nonnegative integer \( n \), \([0]!=1, [n]!= [n][n-1]\cdots [1]\) if \( n>0 \), and let further, \( 0<q<1 \) and

\[
(a + b)_0 = 1, \quad (a + b)_n = (a + b)(a + bq)\cdots (a + bq^{n-1}) \quad \text{if} \quad n>0.
\]

We shall also use the notation

\[
(a)_0 = 1, \quad (a)_n = (1 - a)(1 - aq)\cdots (1 - aq^{n-1}) \quad \text{if} \quad n>0,
\]

and
We first give the following lemmas:

**Lemma 1.** For integers \( n \geq 1 \) and \( m \geq 0 \) we have

\[
A_m = \sum_{j=0}^{m} q^j(q^j - q^{m+1})_{n-1} = \frac{(q^{m+1})_n}{1 - q^n}.
\]

**Proof.** We have

\[
\sum_{m=0}^{\infty} A_m u^m = \sum_{m=0}^{\infty} u^m \sum_{j=0}^{\infty} q^j(q^j - q^{m+1})_{n-1} = \sum_{j=0}^{\infty} u^j q^n \sum_{m=0}^{\infty} (q^{m+1})_{n-1} u^m = \frac{1}{1 - uq^n} \sum_{m=0}^{\infty} \frac{(q)_n}{(q)_m} u^m,
\]

so we have

\[
\sum_{m=0}^{\infty} A_m u^m = \frac{(q)^{n-1}}{1 - uq^n} \sum_{m=0}^{\infty} \frac{(q)_m}{(q)_m} u^m.
\]

Let us recall the formula \([1, \text{p. 66}]\)

\[
\sum_{k=0}^{\infty} \frac{(a)_k}{(q)_k} u^k = \prod_{r=0}^{\infty} \frac{1 - auq^r}{1 - uq^r},
\]

so that (2.2) can now be written as

\[
\sum_{m=0}^{\infty} A_m u^m = \frac{(q)^{n-1}}{1 - uq^n} \prod_{r=0}^{\infty} \frac{1 - uq^{r+n}}{1 - uq^r} = (q)_n \prod_{r=0}^{\infty} \frac{1 - uq^{r+n}}{1 - uq^r} = (q)^{n-1} \sum_{k=0}^{\infty} \frac{(q^{n+1})_k}{(q)_k} u^k.
\]

Equating coefficients of \( u^m \) we get (2.1).

**Lemma 2.** For integral \( m \geq 1 \) we have

\[
\sum_{j=1}^{m} q^{-j}(q^{-m-1} - q^{-j})_{n-1} = \frac{q^{-n(m+1)+1}}{1 - q^n} (q^m)_n.
\]

The proof of this lemma is similar to that of Lemma 1.
Lemma 3. For nonnegative integer $n$,

\[
an^{n+1}(1 - q^n) \sum_{j=0}^{\infty} q^j(q^j - q^{k+1})_{n-1}
\]

(2.4)

\[-ax(1 - q^n) \sum_{j=1}^{\infty} q^j(xq^j - aq^{k+1})_{n-1} - a^{n+1}(q^{k+1})_n
\]

\[= -a(x - aq^{k+1})_n.
\]

Proof. To prove this lemma we recall the identity due to Euler

(2.5)

\[(a + b)_n = \sum_{r=0}^{n} (-1)^r \binom{n}{r} q^{r(r-1)/2} a^{n-r} b^r,
\]

so that

\[
\sum_{j=0}^{\infty} q^j(q^j - q^{k+1})_{n-1} = \sum_{r=0}^{n-1} (-1)^r \binom{n-1}{r} q^{r(r-1)/2} q^{r(k+1)}
\]

\[= a^{n+1} \sum_{r=0}^{n} (-1)^r \binom{n}{r} q^{r(r-1)/2} q^{r(k+1)}
\]

\[= a^{n+1}(1 - q^{k+1})_n - (-1)^n q^{n(n-1)/2+n(k+1)}.
\]

Hence we get

\[
a^{n+1}(1 - q^n) \sum_{j=0}^{\infty} q^j(q^j - q^{k+1})_{n-1}
\]

(2.6)

\[= a^{n+1} \sum_{r=0}^{n-1} (-1)^r \binom{n-1}{r} q^{r(r-1)/2} q^{r(k+1)}
\]

\[= a^{n+1}(1 - q^{k+1})_n - (-1)^n q^{n(n-1)/2+n(k+1)}.
\]

Similarly

\[
ax(1 - q^n) \sum_{j=0}^{\infty} q^j(xq^j - aq^{k+1})_{n-1}
\]

(2.7)

\[= a[(x - aq^{k+1})_n - (-1)^n q^{(1/2)n(n-1)+n(k+1)} a^n].
\]

Substituting (2.6) and (2.7) in the left hand side of (2.4) we get the right hand side of (2.4).

3. We now prove our main results.

Theorem 1. If $n \geq 1$ is a given integer

\[
I^nf(x) = \int_a^x \int_a^{x_{n-1}} \cdots \int_a^{x_1} f(t) d(q, t) d(q, x_1) \cdots d(q, x_{n-1})
\]

(3.1)

\[= \frac{1}{[n-1]!} \int_a^x (x - q^n)_{n-1} f(t) d(q, t),
\]
and

**Theorem 2.** If $n \geq 1$ is an integer

$$K^n f(x) = \int_x^\infty \cdots \int_{x_{n-1}}^\infty f(t) d(q, t) d(q, x_1) \cdots d(q, x_{n-1})$$

(3.2)

$$= \frac{q^{-(1/2)n(n-1)}}{[n - 1]!} \int_x^\infty (t - x)_{n-1} f(tq^{1-n}) d(q, t).$$

Both of these theorems can be proved by induction. We give the proof of the first as the proof of the second is similar. To prove Theorem 1 assume (3.1) is true for $n = N$. We have

$$I^{N+1} f(x) = \frac{1 - q}{[N - 1]!} \int_x^\infty \sum_{k=0}^\infty q^k \{ t(t - tq^{k+1})_{N-1} f(tq^k)$$

$$- a(t - aq^{k+1})_{N-1} f(aq^k) \} d(q, t)$$

$$= \frac{(1 - q)^2}{[N - 1]!} \sum_{k=0}^\infty \sum_{j=0}^\infty q^{k+j} (1 - q^{k+1})_{N-1} \{ x^{N+1} q^N f(xq^{k+j})$$

$$- a^{N+1} q^N f(aq^{k+j}) \}$$

$$- \frac{(1 - q)^2}{[N - 1]} \sum_{k=0}^\infty \sum_{j=0}^\infty q^{k+j} f(aq^k) \{ ax(xq^j - aq^{k+1})_{N-1}$$

$$- a^{N+1} (q^j - q^{k+1})_{N-1} \}.$$

This reduces after some simplification to

$$[N - 1]! I^{N+1} f(x) = (1 - q)^2 \sum_{s=0}^\infty q^s f(xq^s) (x^{N+1} - a^{N+1}) \sum_{j=0}^\infty q^j (q^j - q^{s+1})_{N-1}$$

$$- (1 - q)^2 \sum_{s=0}^\infty q^s f(xq^s) ax \sum_{j=0}^\infty q^j (xq^j - aq^{s+1})_{N-1}$$

$$+ (1 - q^2) \sum_{s=0}^\infty q^s f(xq^s) a^{N+1} \sum_{j=0}^\infty q^j (q^j - q^{s+1})_{N-1}.$$

Evaluating the inside sums in the right hand side of the above equation by means of Lemmas 1 and 3 we get

$$I^{N+1} f(x) = \frac{(1 - q)}{[N]!} \sum_{s=0}^\infty q^s \{ x(x - xq^{s+1})_{Nf}(xq^s) - a(x - aq^{s+1})_{Nf}(aq^s) \}$$

$$= \frac{1}{[N]!} \int_a^x (x - qt)_{Nf} f(t) d(q, t).$$
Since (3.1) is true for $n=1$ our proof is complete. We remark that Lemma 2 is required in the proof of Theorem 2.

REFERENCES


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