SOME RELATIONS ASSOCIATED WITH AN EXTENSION OF KOSHLI AKOV'S FORMULA

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It is known that the following three relations are equivalent:

1. Functional equation for $\xi^2(s)$

\[ \pi^{-s} \left[ \Gamma \left( \frac{s}{2} \right) \right]^2 \xi^2(s) = \pi^{s-1} \left[ \Gamma \left( \frac{1-s}{2} \right) \right]^2 \left[ \xi(1-s) \right]^2; \]

2. Koshliakov's formula

\[ \gamma - \log \left( \frac{4\pi}{\tau} \right) + 4 \sum_{n=1}^{\infty} d(n) K_0(2\pi \tau n) \]

\[ = \tau^{-1} \left[ \gamma - \log(4\pi\tau) \right] + 4\tau^{-1} \sum_{n=1}^{\infty} d(n) K_0 \left( \frac{2\pi}{\tau} n \right); \]

3. Voronoi's sum formula

\[ - \frac{1}{4} f(0) + \sum_{n=1}^{\infty} d(n) f(n) = \int_{0}^{\infty} (2\gamma + \log x) f(x) \, dx \]

\[ + 4 \sum_{n=1}^{\infty} d(n) \int_{0}^{\infty} f(x) \left[ K_0(4\pi(nx)^{1/2}) - \frac{\pi}{2} V_0(4\pi(nx)^{1/2}) \right] \, dx. \]

In fact, these were obtained in [5] by specializing parameters in relations which were proved equivalent. In the present note, we give a proof of the statement [5, p. 63] that two other special cases of (3), namely

\[ \frac{\rho}{2\pi^2} \sum_{n=1}^{\infty} d(n) \log \left( \frac{n^2}{\rho} \right) \]

\[ \gamma \cdot \frac{1}{4} + \frac{1}{8} \log \rho + \frac{1}{8\pi^2 \rho} \log (4\pi^2 \rho) \]

\[ + \sum_{n=1}^{\infty} d(n) K_0(4\pi(n\rho)^{1/2}) \]

and

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\[ \rho \sum_{n=1}^{\infty} \frac{d(n)}{\rho^2 + n^2} = \frac{1}{4\rho} + \gamma \pi + \frac{1}{2} \pi \log \rho \]

(5)

\[ + 2\pi \sum_{n=1}^{\infty} d(n) \left\{ K_0(4\pi (-in\rho)^{1/2}) + K_0(4\pi (in\rho)^{1/2}) \right\} \]

can also be considered equivalent to (3). Thus any one of the relations (1) through (5) implies the others. However, as in [5], we shall treat the problem in a somewhat generalized form.

In the course of our investigation, we shall need the following results:

(6) \[ \int_{0}^{\infty} x^{p-1} K_0(xy) \, dx = \frac{1}{4} \left( \frac{2}{y} \right)^p \left[ \Gamma \left( \frac{p}{2} \right) \right]^2, \quad [1, p. 127], \]

(7) \[ \int_{0}^{\infty} K_0 \left( \frac{a}{x} \right) K_0(xy) \, dx = \frac{\pi}{y} K_0(2ay^{1/2}), \quad [1, p. 146], \]

(8) \[ \int_{0}^{\infty} xK_0(ax)K_0(xy) \, dx = \frac{\log \left( \frac{a}{y} \right)}{a^2 - y^2}, \]

\[ \int_{0}^{\infty} \log x \, x^{p-1} K_0(xy) \, dx \]

(9) \[ = \frac{1}{4} \left( \frac{2}{y} \right)^p \left[ \Gamma \left( \frac{p}{2} \right) \right]^2 \left\{ \log \left( \frac{2}{y} \right) + \psi \left( \frac{p}{2} \right) \right\}. \]

(8) can easily be obtained from [1, p. 145] and (9) from (6) above.

(9a) \[ \int_{0}^{\infty} K_0(ax) \left[ \frac{2}{\pi} K_0((xy)^{1/2}) - Y_0((xy)^{1/2}) \right] \, dx = \frac{1}{a} K_0 \left( \frac{y}{4a} \right) \]

[5, p. 51].

1. We assume that

- \[ a_1, a_2 \cdots; \quad 0 < \lambda_1 < \lambda_2 \cdots \rightarrow \infty, \]

- \[ b_1, b_2 \cdots; \quad 0 < l_1 < l_2 \cdots \rightarrow \infty \]

are four sequences of numbers such that the series

\[ \sum_{n=1}^{\infty} \frac{|a_n|}{\lambda_n^2} \quad \text{and} \quad \sum_{n=1}^{\infty} \frac{|b_n|}{l_n^2} \]

are convergent.

(b) \[ \alpha, \alpha', \beta, \beta' \] are four numbers such that the following relation holds
\[
\alpha[\gamma + \log(\pi \tau)] - \alpha' + \sum_{n=1}^{\infty} a_n K_0(2\pi \sqrt{n} \tau) \\
= \frac{1}{\tau} \left\{ \beta \left[ \gamma + \log \left( \frac{\pi}{\tau} \right) \right] - \beta' + \sum_{n=1}^{\infty} b_n K_0 \left( \frac{2\pi \sqrt{n}}{\tau} \right) \right\}.
\]

If we multiply both sides of (I) by \( \tau K_0(2\pi \tau y) \), \((y > 0)\), and integrate with regard to \( \tau \) from zero to infinity, we get

\[
\frac{1}{4\pi^2 y^2} \left[ \alpha \gamma + \alpha \log \pi - \alpha' \right] + \frac{1}{4\pi^2 y^2} \left[ -\alpha \log \pi - \gamma \right] + \frac{1}{4\pi^2} \sum_{n=1}^{\infty} a_n \frac{\log \left( \frac{\lambda_n}{y} \right)}{\lambda_n^2 - y^2}
\]

\[
= \left[ \beta \gamma + \beta \log \pi - \beta' \right] \frac{1}{4y} + \beta \left[ \gamma + \log (4\pi y) \right] + \frac{1}{2y} \sum_{n=1}^{\infty} b_n K_0(4\pi (l_n y)^{1/2}).
\]

The interchange of the order of summation and integration is justified because of absolute convergence.

We rewrite (10) as

\[
\frac{1}{2\pi^2 y} \left[ -\alpha \log y - \alpha' \right] + \frac{y}{2\pi^2} \sum_{n=1}^{\infty} a_n \frac{\log y}{\lambda_n^2 - y^2}
\]

\[
= \left[ \beta \gamma + \frac{1}{2} \beta \log (4\pi^2 y) - \frac{1}{2} \beta' \right] + \sum_{n=1}^{\infty} b_n K_0(4\pi (l_n y)^{1/2}).
\]

By analytic continuation (II) holds for all \( y \) such that \(-\pi < \arg y < \pi\). Likewise, if we multiply (I) by \((1/\tau^2)K_0(2\pi y/\tau)\) and integrate, we get

\[
\left( \alpha \gamma + \frac{1}{2} \alpha \log (4\pi^2 y) - \frac{1}{2} \alpha' \right) + \sum_{n=1}^{\infty} a_n K_0(4\pi (\lambda_n y)^{1/2})
\]

\[
= \frac{1}{2\pi^2 y} \left( -\beta \log y - \beta' \right) + \frac{y}{2\pi^2} \sum_{n=1}^{\infty} b_n \frac{\log \left( \frac{l_n}{y} \right)}{l_n^2 - y^2}.
\]

If we replace \( y \) by \( e^{i\pi y} \) in (II), we get the relation
\[
\frac{1}{2\pi^2y} \left[ \alpha \log \left( ye^{ix} \right) + \alpha' \right] + \frac{y}{2\pi^2} \sum_{n=1}^{\infty} a_n \frac{\log \left( \frac{\lambda_n}{ye^{ix}} \right)}{y^2 - \lambda_n^2}
\]

(11)

\[
= \left[ \beta \gamma + \frac{1}{2} \beta \log \left( 4\pi^2 ye^{ix} \right) - \frac{1}{2} \beta' \right] + \sum_{n=1}^{\infty} b_n K_0(4\pi(l_n ye^{ix})^{1/2}), 
-2\pi < \arg y < 0.
\]

Adding the corresponding sides of (II) and (11), we obtain

\[
\frac{i\alpha}{2\pi y} - \frac{iy}{2\pi} \sum_{n=1}^{\infty} \frac{a_n}{y^2 - \lambda_n^2}
\]

(III) \[
= 2\beta \gamma + \beta \log \left( 4\pi^2 y \right) - \beta' + \frac{1}{2}i\pi\beta
\]

\[
+ \sum_{n=1}^{\infty} b_n \left\{ K_0 \left( 4\pi(l_n y)^{1/2} \right) + K_0 \left( 4\pi(l_n ye^{ix})^{1/2} \right) \right\}, 
-\pi < \arg y < 0.
\]

The corresponding relation derived from (II') is

\[
2\alpha \gamma + \alpha \log \left( 4\pi^2 y \right) - \alpha' + \frac{1}{2}i\pi\alpha
\]

(III') \[
+ \sum_{n=1}^{\infty} a_n \left\{ K_0 \left( 4\pi(\lambda_n y)^{1/2} \right) + K_0 \left( 4\pi(\lambda_n ye^{ix})^{1/2} \right) \right\}
\]

\[
= \frac{i\beta}{2\pi y} - \frac{iy}{2\pi} \sum_{n=1}^{\infty} \frac{b_n}{y^2 - l_n^2}, 
-\pi < \arg y < 0.
\]

2. Let

\[
\sigma(y) = \frac{1}{2\pi i} \left\{ \sum_{n=1}^{\infty} a_n \frac{2y}{y^2 - \lambda_n^2} - 2\frac{\alpha}{\gamma} \right\}.
\]

Then from (III)

\[
\frac{1}{2} \sigma(y) = 2\beta \gamma + \beta \log \left( 4\pi^2 y \right) - \beta' + \frac{1}{2}i\pi\beta
\]

\[
+ \sum_{n=1}^{\infty} b_n \left\{ K_0 \left( 4\pi(l_n y)^{1/2} \right) + K_0 \left( 4\pi(l_n ye^{ix})^{1/2} \right) \right\}, 
-\pi < \arg y < 0,
\]

\[
= - \left[ 2\beta \gamma + \beta \log \left( 4\pi^2 y \right) - \beta' \right] + \frac{1}{2}i\pi\beta
\]

\[
- \sum_{n=1}^{\infty} b_n \left\{ K_0 \left( 4\pi(l_n y)^{1/2} \right) + K_0 \left( 4\pi(l_n ye^{-ix})^{1/2} \right) \right\}, 
0 < \arg y < \pi.
\]

Let \(\phi(y)\) be a function of the complex variable \(y = u + iv\), regular in a strip \(u \geq 0, |v| \leq \delta\), for some \(\delta > 0\) and satisfying the following conditions:
(i) \( \int_0^\infty |\phi(u+iv)|\,du \) and \( \int_0^\infty \log (u+iv) \phi(u+iv)\,du \) converge in 
\( -\delta < v < \delta. \)

(ii) There exists a sequence of the numbers \( u_n \) such that

\[ \lim_{n \to \infty} (u_n + iv)\phi(u_n + iv) = 0 \]

uniformly in \( -\delta < v < \delta. \)

Since \( \sigma(y) \) is analytic except for simple poles at zero and \( \pm \lambda_n, n=1, 2, 3, \ldots \), we obtain by Cauchy's theorem and (13)

\[ \sum_{n=1}^\infty a_n\phi(\lambda_n) = \int_C \sigma(y)\phi(y)\,dy \]

where \( C \) is the contour shown. Now we let \( \varepsilon \to 0. \)

Thus

\[ \sum_{n=1}^\infty a_n\phi(\lambda_n) \]

\[ = a\phi(0) + 4\int_0^\infty [2\beta y - \beta' + \beta \log (4\pi^2 u)]\phi(u)\,du \]

\[ + \lim_{\varepsilon \to 0} \left\{ 2\int_{-\varepsilon}^{\varepsilon} \phi(y) \sum_{n=1}^\infty b_n [K_0(4\pi(l_n y)^{1/2}) + K_0(4\pi(l_n y e^{i\pi})^{1/2})] \,dy \right\} \cdot \]

\[ + 2\int_{\varepsilon}^{\infty+\varepsilon} \phi(y) \sum_{n=1}^\infty b_n [K_0(4\pi(l_n y)^{1/2}) + K_0(4\pi(l_n y e^{-i\pi})^{1/2})] \,dy \}

Let \( \phi(y) \) be such that we can interchange the order of summation and integration. Then proceeding to the limit as \( \varepsilon \to 0; \) and using

\[ K_0(\text{ye}^{i\pi/2}) + K_0(\text{ye}^{-i\pi/2}) = -\pi Y_0(y), \]

we obtain
\[-\alpha \phi(0) + \sum_{n=1}^{\infty} a_n \phi(\lambda_n) = 4 \int_0^\infty \left[ 2\beta \gamma - \beta' + \beta \log (4\pi^2u) \right] \phi(u) \, du \]

(IV)

\[+ 4 \sum_{n=1}^{\infty} b_n \int_0^\infty \left[ K_0(4\pi(l_nu)^{1/2}) - \frac{\pi}{2} Y_0(4\pi(l_nu)^{1/2}) \right] \phi(u) \, du.\]

Likewise from (III'), we get the Sum-formula

\[-\beta \phi(0) + \sum_{n=1}^{\infty} b_n \phi(l_n) = 4 \int_0^\infty \left[ 2\alpha \gamma - \alpha' + \alpha \log (4\pi^2u) \right] \phi(u) \, du \]

(IV')

\[+ 4 \sum_{n=1}^{\infty} a_n \int_0^\infty \left[ K_0(4\pi(\lambda_nu)^{1/2}) - \frac{\pi}{2} Y_0(4\pi(\lambda_nu)^{1/2}) \right] \phi(u) \, du.\]

If in (IV) or (IV'), we let

\[\phi(u) = K_0(2\pi ur) - K_0(2\pi u), \quad 0 < u < \infty,\]

\[\phi(0) = \log \frac{1}{r},\]

and further assume that

\[\alpha[\gamma + \log \pi] - \alpha' + \sum_{n=1}^{\infty} a_n K_0(2\pi \lambda_n)\]

(c)

\[= \beta[\gamma + \log \pi] - \beta' + \sum_{n=1}^{\infty} b_n K_0(2\pi l_n).\]

Then using (6), (9), (9a), we get the relation (I).

Thus we have proved the following:

1. Under condition (a), each one of the six relations (II), (II'), (III), (III'), (IV), (IV') is a consequence of (I).

2. Under conditions (a) and (c), the relation (I) through (IV') are equivalent to each other.

If we set

\[a_n = b_n = d(n); \quad \lambda_n = l_n = n,\]

\[\alpha = \beta = \frac{1}{4}; \quad \alpha' = \beta' = \frac{1}{2} \log (2\pi),\]

in relations (I)-(IV') and note that conditions (a) and (c) are satisfied, we get the corresponding special cases mentioned in the introduction.

3. Now we study the functions \(f(s)\) and \(g(s)\) defined by:
When conditions (a) and (b) hold.

Let

\[ \mu(y) = \sum_{n=1}^{\infty} b_n \left\{ K_0(4\pi(l_n y)^{1/2}) + K_0(4\pi(l_n y e^{i\pi})^{1/2}) \right\}, \quad -\pi < \arg y < 0, \]

then

\[ \mu(e^{-i\pi} y) = \sum_{n=1}^{\infty} b_n \left\{ K_0(4\pi(l_n y e^{-i\pi})^{1/2}) + K_0(4\pi(l_n y)^{1/2}) \right\}, \quad 0 < \arg y < \pi. \]

We denote the contour opposite as \( C_r \).
\[ f(s) = 4\left[2\beta \gamma - \beta' + \beta \log(4\pi^2)\right] \frac{\nu^{1-s}}{s-1} + 4\beta \log \nu \frac{\nu^{1-s}}{s-1} \]

\[ + 4\beta \frac{\nu^{1-s}}{(s-1)^2} + 2 \int_{-i\infty}^{i\infty} \frac{\mu(y)}{y^s} \, dy + 2 \int_{y+i\infty}^{y+i\infty} \frac{\mu(e^{-i\gamma}y)}{y} \, dy. \quad (14) \]

But the above integrals define a function of \( s \) which is regular for all finite values of \( s \). Therefore \( f(s) \) is analytic and single valued in the whole \( s \)-plane except perhaps at \( s = 1 \), where it may have a pole of the second order, with the principal part

\[ \frac{8\beta \gamma - 4\beta' + 4\beta \log(4\pi^2)}{s-1} + \frac{4\beta}{(s-1)^2}. \]

Likewise, using (III'), which is a consequence of (b), we can show that \( g(s) \) has the same properties as \( f(s) \), and that the principal part of \( g(s) \) at \( s = 1 \) is

\[ \frac{4\alpha}{(s-1)^2} + \frac{8\alpha \gamma - 4\alpha' + 4\alpha \log(4\pi^2)}{s-1}. \]

Now if we proceed to the limit as \( \nu \to 0 \) in \( \Re s < 1 \), we obtain from (14)

\[ f(s) = 2 \int_0^{-i\infty} \frac{\mu(y)}{y^s} \, dy + 2 \int_0^{i\infty} \frac{\mu(e^{-i\gamma}y)}{y^s} \, dy \]

\[ = 4 \sin\left(\frac{\pi}{2} s\right) \int_0^\infty \frac{\mu(e^{-i\gamma/2}t)}{t^s} \, dt. \quad (15) \]

This gives the integral representation of \( f(s) \) in the half \( s \)-plane.

If we interchange the order of integration and summation in (15) and use

\[ \int_0^\infty K_0(at^{1/2})t^{-s} \, dt = a^{2s-2}2^{1-2s} \Gamma(1-s)^2, \quad \Re s < 1; \quad |\arg a| < \frac{\pi}{2}, \]

we obtain

\[ f(s) = \left[ \sin\left(\frac{\pi}{2} s\right) \right]^2 \Gamma(1-s)^2 2^{2s-2}g(1-s)^{2s-2}, \quad \Re s < -1, \]

which can be written as

\[ \pi^{-s} \left[ \Gamma\left(\frac{s}{2}\right) \right]^2 f(s) = \pi^{-1+s} \left[ \Gamma\left(\frac{1-s}{2}\right) \right]^2 g(1-s). \quad (16) \]
and by analytic continuation equality (16) holds for all $s$.

Finally, expanding both sides of (16) around $s = 0$ and $s = 1$ respectively, we find that

\begin{align}
\alpha &= f(0), \quad \alpha' = f'(0), \quad \beta = g(0), \quad \beta' = g'(0).
\end{align}

Thus we have proved the following: Under conditions (a) and (b), $(s - 1)^2 f(s)$ and $(s - 1)^2 g(s)$ are entire functions; $f(s)$ and $g(s)$ satisfy the functional equation (16) and further $\alpha$, $\alpha'$, $\beta$, $\beta'$ are related to these functions as given in (17).

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REFERENCES


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