A COMPARISON THEOREM FOR ELLIPTIC DIFFERENTIAL EQUATIONS

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Clark and the author [2] recently obtained a generalization of the Hartman-Wintner comparison theorem [4] for a pair of self-adjoint second order linear elliptic differential equations. The purpose of this note is to extend this generalization to general second order linear elliptic equations. As in [2], the usual pointwise inequalities for the coefficients are replaced by a more general integral inequality. The result is new even in the one-dimensional case, and extends Leighton’s result for self-adjoint ordinary equations [5].

Protter [6] obtained pointwise inequalities in the nonself-adjoint case in two dimensions by the method of Hartman and Wintner [4]. We obtain an alternative to Protter’s result as a corollary of our main theorem.

Let $R$ be a bounded domain in $n$-dimensional Euclidean space with boundary $B$ having a piecewise continuous unit normal. The linear elliptic differential operator $L$ defined by

$$Lu = \sum_{i,j=1}^{n} D_i(a_{ij}D_ju) + 2\sum_{i=1}^{n} b_iD_iu + cu, \quad a_{ij} = a_{ji}$$

will be considered in $R$, where $D_i$ denotes partial differentiation with respect to $x^i, i = 1, 2, \cdots, n$. We assume that the coefficients $a_{ij}, b_i,$ and $c$ are real and continuous on $\bar{R}$, the $b_i$ are differentiable in $R$, and that the symmetric matrix $(a_{ij})$ is positive definite in $R$. A “solution” $u$ of $Lu = 0$ is supposed to be continuous on $\bar{R}$ and have uniformly continuous first partial derivatives in $R$, and all partial derivatives involved in (1) are supposed to exist, be continuous, and satisfy $Lu = 0$ in $R$.

Let $Q[z]$ be the quadratic form in $(n+1)$ variables $z_1, z_2, \cdots, z_{n+1}$ defined by

$$Q[z] = \sum_{i,j=1}^{n} a_{ij}z_i z_j - 2z_{n+1} \sum_{i=1}^{n} b_i z_i + g z_{n+1}^2,$$

where the continuous function $g$ is to be determined so that this form is positive semidefinite. The matrix $Q$ associated with $Q[z]$ has the block form

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\[
Q = \begin{pmatrix} A & -b \\ -b^T & g \end{pmatrix}, \quad A = (a_{ij}),
\]

where \(b^T\) is the \(n\)-vector \((b_1, b_2, \ldots, b_n)\). Let \(B_i\) denote the cofactor of \(-b_i\) in \(Q\). Since \(A\) is positive definite, a necessary and sufficient condition for \(Q\) to be positive semidefinite is \(\det Q \geq 0\), or

\[
g \det (a_{ij}) \geq - \sum_{i=1}^{n} b_i B_i.
\]

The proof is a slight modification of the well-known proof for positive definite matrices [3, p. 306].

Let \(J\) be the quadratic functional defined by

\[
J[u] = \int_{\mathbb{R}} F[u] \, dx
\]

where

\[
F[u] = \sum_{i,j} a_{ij} D_i u D_j u - 2u \sum_i b_i D_i u + (g - c)u^2,
\]

with domain \(\mathcal{D}\) consisting of all real-valued continuous functions on \(\mathbb{R}\) which have uniformly continuous first partial derivatives in \(\mathbb{R}\) and vanish on \(B\).

**Lemma.** Suppose \(g\) satisfies (3). If there exists \(u \in \mathcal{D}\) not identically zero such that \(J[u] < 0\), then every solution \(v\) of \(Lv = 0\) vanishes at some point of \(\mathbb{R}\).

**Proof.** Suppose to the contrary that there exists a solution \(v \neq 0\) in \(\mathbb{R}\). For \(u \in \mathcal{D}\) define

\[
X^i = v D_i (u/v);
\]

\[
Y^i = v^{-1} \sum_j a_{ij} D_j v, \quad i = 1, 2, \ldots, n;
\]

\[
E[u, v] = \sum_{i,j} a_{ij} X^i X^j - 2u \sum_i b_i X^i + gu^2 + \sum_i D_i(u^2 Y^i).
\]

A routine calculation yields the identity

\[
\]

Since \(Lv = 0\) in \(\mathbb{R}\),
\[ J[u] = \int_R \left[ \sum_{i,j} a_{ij}X^iX^j - 2u \sum_i b_iX^i + gu^2 \right] dx \]

(5)

\[ + \int_R \sum_i D_i(u^2V) \, dx. \]

Since \( u = 0 \) on \( B \), the second integral is zero by Green's formula. The first integrand is a positive semidefinite form by hypothesis (3). The contradiction \( J[u] \geq 0 \) establishes the lemma.

Consider in addition to (1) a second differential operator \( L^* \) of the same form,

\[ L^* u = \sum_{i,j=1}^n D_i(a^*_{ij}D_ju) + 2 \sum_i b^*_i D_iu + c^* u, \quad a^*_{ij} = a_{jj} \]

in which the coefficients satisfy the same conditions as the coefficients in (1). \( L^* \) is the Euler-Jacobi operator associated with the quadratic functional \( J^* \) defined by

\[ J^*[u] = \int_R \left[ \sum_{i,j} a^*_{ij}D_iuD_ju - 2u \sum_i b^*_i D_iu - c^*u^2 \right] dx. \]

(6)

Define \( V[u] = J^*[u] - J[u], \ u \in \Omega \). Since \( u = 0 \) on \( B \), it follows from partial integration that

\[ V[u] = \int_R \left[ \sum (a^*_{ij} - a_{ij}) D_iuD_ju \right. \]

\[ + \left\{ \sum D_i(b^*_i - b_i) + c - c^* - g \right\} u^2 \] \, dx.

(7)

**Theorem 1.** Suppose \( g \) satisfies (3). If there exists a nontrivial solution \( u \) of \( L^*u = 0 \) in \( R \) such that \( u = 0 \) on \( B \) and \( V[u] > 0 \), then every solution of \( Lv = 0 \) vanishes at some point of \( \overline{R} \).

**Proof.** The hypothesis \( V[u] > 0 \) is equivalent to \( J[u] < J^*[u] \). Since \( u = 0 \) on \( B \), it follows from Green's formula that \( J^*[u] = 0 \). Hence the hypothesis \( J[u] < 0 \) of the lemma is fulfilled.

**Theorem 2.** Suppose \( g \) det \((a_{ij}) > -\sum b_iB_i \). If there exists a nontrivial solution of \( L^*u = 0 \) in \( R \) such that \( u = 0 \) on \( B \) and \( V[u] \geq 0 \), then every solution of \( Lv = 0 \) vanishes at some point of \( \overline{R} \).

Since \( Q \) is positive definite, the lemma is valid when the hypothesis \( J[u] < 0 \) is replaced by \( J[u] \leq 0 \). The proof of Theorem 2 is then analogous to that of Theorem 1.
In the case that equality holds in (3), that is

\[ g = - \sum_i b_i B_i / \det (a_{ij}), \]

define

\[ \delta = \sum D_i (b_i^* - b_i) + c - c^* - g. \]

\( L \) is called a "strict Sturmian majorant" of \( L^* \) by Hartman and Wintner [4] when the following conditions hold: (i) \( (a_i^* - a_{ij}) \) is positive semidefinite and \( \delta \geq 0 \) in \( R \); (ii) either \( \delta > 0 \) at some point or \( (a_i^* - a_{ij}) \) is positive definite and \( c^* \neq 0 \) at some point. The corollary below follows immediately from Theorem 1.

**Corollary.** Suppose that \( L \) is a strict Sturmian majorant of \( L^* \). If there exists a solution \( u \) of \( L^* u = 0 \) in \( R \) such that \( u = 0 \) on \( B \) and \( u \) does not vanish in any open set contained in \( R \), then every solution of \( L v = 0 \) vanishes at some point of \( R \).

If the coefficients \( a_{ij}^* \) are of class \( C^{2,1}(R) \) (i.e. all second derivatives exist and are Lipschitzian), the hypothesis that \( u \) does not vanish in any open set of \( R \) can be replaced by the condition that \( u \) does not vanish identically in \( R \) because of Aronszajn's unique continuation theorem [1].

In the case \( n = 2 \) considered by Protter [6], the condition \( \delta \geq 0 \) reduces to

\[ (a_{11}a_{22} - a_{12}^2) \left( \sum_{i=1}^2 D_i (b_i^* - b_i) + c - c^* \right) \]

\[ \geq a_{11}b_2^2 - 2a_{12}b_1b_2 + a_{22}b_1^2, \]

which is considerably simpler than Protter's condition. It reduces to Protter's condition

\[ \sum_{i=1}^2 D_i b_i^* + c - c^* \geq 0 \]

in the case that \( b_1 = b_2 = 0 \), and also in the case that \( a_{12} = a_{12}^* = 0, a_{11} = a_{11}^*, a_{22} = a_{22}^* \). (Two incorrect signs appear in [6]).

The following example in the case \( n = 2 \) illustrates that Theorem 1 is more general than the pointwise condition (9). Let \( R \) be the square \( 0 < x^1, x^2 < \pi \). Let \( L^*, L \) be the elliptic operators defined by
\[ L^*u = D_1^2u + D_2^2u + 2u, \]
\[ L_v = D_1^2v + D_2^2v + D_1v + cv, \]
where
\[ c(x^1, x^2) = f(x^1)f(x^2) + 5/4, \]
and \( f \in C[0, \pi] \) is not identically zero. The function \( u = \sin x^1 \sin x^2 \) is zero on \( B \) and satisfies \( L^*u = 0 \). The condition \( V[u] > 0 \) of Theorem 1 reduces to
\[ \int_0^\pi \int_0^\pi f(x^1)f(x^2) \sin^2 x^1 \sin^2 x^2 dx^1 dx^2 > 0. \]
Since this is fulfilled, every solution of \( L_v = 0 \) vanishes at some point of \( \overline{R} \). This cannot be concluded from (9) or from Protter's result [6] unless \( f \) has constant sign in \( R \).

In the case \( n = 1 \), \( L \) is an ordinary differential operator of the form
\[ Lu = (au')' + 2bu' + cu, \]
and \( R \) is an interval \((x_1, x_2)\). We assert that \( \overline{R} \) can be replaced by \( R \) in the lemma and theorems; for \( v \) can have at most a simple zero at the boundary points \( x_1 \) and \( x_2 \), and hence the first integral on the right side of (5) exists and is nonnegative provided only that \( v \neq 0 \) in \( R \).

**Theorem 3.** If there exists a nontrivial solution \( u \) of \( L^*u = 0 \) in \( R \) such that \( u = 0 \) on \( B \) and
\[ \int_{x_1}^{x_2} \left[ (a^* - a)u'^2 + \left( b^* - b' + c - c^* - \frac{b^{2^2}}{a} \right) u^2 \right] dx > 0, \]
then every solution of \( L_v = 0 \) has a zero in \((x_1, x_2)\).

In the self-adjoint case \( b = b^* = 0 \) it was shown by Clark and the author [2] that the strict inequality in the hypothesis \( V[u] > 0 \) of Theorem 1, and therefore also in (10), can be replaced by \( \geq \). Indeed, this is transparent when the proof of the above lemma is specialized to the self-adjoint case. With \( > \) replaced by \( \geq \), (10) reduces to Leighton's condition in the self-adjoint case [5].

**References**


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*q-ANALOGUES OF CAUCHY'S FORMULAS*

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1. Let $q$ be a given number and let $\alpha$ be real or complex. The $\alpha$th "basic number" is defined by means of $[\alpha] = (1-q^\alpha)/(1-q)$. This has served as a basis for an extensive amount of literature in mathematics under such titles as Heine, basic, or $q$-series and functions. The basic numbers also occur naturally in many theta identities. The works of Jackson (for bibliography see [2]) and Hahn [3] have stimulated much interest in this field.

One important operation that is intimately connected with basic series as well as with difference and other functional equations is the $q$-derivative of a function $f$. This is defined by

\begin{equation}
Df(x) = \frac{f(qx) - f(x)}{x(q - 1)}.
\end{equation}

Jackson defined the operations, which he called $q$-integration,

\begin{equation}
\int_0^x f(t)\,d(q, t) = x(1 - q) \sum_{k=0}^{\infty} q^k f(xq^k)
\end{equation}

and

\begin{equation}
\int_x^\infty f(t)\,d(q, t) = x(1 - q) \sum_{k=0}^{\infty} q^{-k} f(xq^{-k})
\end{equation}

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