A COMPARISON THEOREM FOR ELLIPTIC DIFFERENTIAL EQUATIONS

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Clark and the author [2] recently obtained a generalization of the Hartman-Wintner comparison theorem [4] for a pair of self-adjoint second order linear elliptic differential equations. The purpose of this note is to extend this generalization to general second order linear elliptic equations. As in [2], the usual pointwise inequalities for the coefficients are replaced by a more general integral inequality. The result is new even in the one-dimensional case, and extends Leighton’s result for self-adjoint ordinary equations [5].

Protter [6] obtained pointwise inequalities in the nonself-adjoint case in two dimensions by the method of Hartman and Wintner [4]. We obtain an alternative to Protter’s result as a corollary of our main theorem.

Let \( \mathcal{R} \) be a bounded domain in \( n \)-dimensional Euclidean space with boundary \( \partial \mathcal{R} \) having a piecewise continuous unit normal. The linear elliptic differential operator \( L \) defined by

\[
Lu = \sum_{i,j=1}^{n} D_i(a_{ij}D_ju) + 2 \sum_{i=1}^{n} b_i D_iu + cu, \quad a_{ij} = a_{ji}
\]

will be considered in \( \mathcal{R} \), where \( D_i \) denotes partial differentiation with respect to \( x^i, i = 1, 2, \ldots, n \). We assume that the coefficients \( a_{ij}, b_i, \) and \( c \) are real and continuous on \( \overline{\mathcal{R}} \), the \( b_i \) are differentiable in \( \mathcal{R} \), and that the symmetric matrix \( (a_{ij}) \) is positive definite in \( \mathcal{R} \). A “solution” \( u \) of \( Lu = 0 \) is supposed to be continuous on \( \overline{\mathcal{R}} \) and have uniformly continuous first partial derivatives in \( \mathcal{R} \), and all partial derivatives involved in (1) are supposed to exist, be continuous, and satisfy \( Lu = 0 \) in \( \mathcal{R} \).

Let \( Q[z] \) be the quadratic form in \( (n+1) \) variables \( z_1, z_2, \ldots, z_{n+1} \) defined by

\[
Q[z] = \sum_{i,j=1}^{n} a_{ij}z_i z_j - 2 \sum_{i=1}^{n} b_i z_i + g z_{n+1}^2,
\]

where the continuous function \( g \) is to be determined so that this form is positive semidefinite. The matrix \( Q \) associated with \( Q[z] \) has the block form

Received by the editors October 21, 1965.

1 This research was supported by the United States Air Force Office of Scientific Research, under grant AF-AFOSR-379-65.

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\[ Q = \begin{pmatrix} A & -b \\ -b^T & g \end{pmatrix}, \quad A = (a_{ij}), \]

where \( b^T \) is the \( n \)-vector \((b_1, b_2, \ldots, b_n)\). Let \( B_i \) denote the cofactor of \(-b_i\) in \( Q \). Since \( A \) is positive definite, a necessary and sufficient condition for \( Q \) to be positive semidefinite is \( \det Q \geq 0 \), or

\[ g \det (a_{ij}) \geq - \sum_{i=1}^{n} b_i B_i. \tag{3} \]

The proof is a slight modification of the well-known proof for positive definite matrices [3, p. 306].

Let \( J \) be the quadratic functional defined by

\[ J[u] = \int_{\mathcal{D}} F[u] \, dx \tag{4} \]

where

\[ F[u] = \sum_{i,j} a_{ij} D_iu D_ju - 2u \sum_{i} b_i D_iu + (g - c)u^2, \]

with domain \( \mathcal{D} \) consisting of all real-valued continuous functions on \( \mathcal{R} \) which have uniformly continuous first partial derivatives in \( \mathcal{R} \) and vanish on \( \mathcal{B} \).

**Lemma.** Suppose \( g \) satisfies (3). If there exists \( u \in \mathcal{D} \) not identically zero such that \( J[u] < 0 \), then every solution \( v \) of \( Lv = 0 \) vanishes at some point of \( \mathcal{R} \).

**Proof.** Suppose to the contrary that there exists a solution \( v \neq 0 \) in \( \mathcal{R} \). For \( u \in \mathcal{D} \) define

\[ X^i = v D_i (u/v) \]

\[ Y^i = v^{-1} \sum_{j} a_{ij} D_j v, \quad i = 1, 2, \ldots, n; \]

\[ E[u, v] = \sum_{i,j} a_{ij} X^i X^j - 2u \sum_{i} b_i X^i + gu^2 + \sum_{i} D_i(u^2 Y^i). \]

A routine calculation yields the identity

\[ E[u, v] = F[u] + u^2 v^{-1} Lv. \]

Since \( Lv = 0 \) in \( \mathcal{R} \),
\[ J[u] = \int_R \left[ \sum_{i,j} a_{ij}X_iX_j - 2u \sum_i b_i X_i + gu^2 \right] dx \]
\[ + \int_R \sum_i D_i(u^2Y^i) \, dx. \]

Since \( u = 0 \) on \( B \), the second integral is zero by Green's formula. The first integrand is a positive semidefinite form by hypothesis (3). The contradiction \( J[u] \geq 0 \) establishes the lemma.

Consider in addition to (1) a second differential operator \( L^* \) of the same form,
\[ L^* u = \sum_{i,j=1}^n D_i(a_{ij}^* D_j u) + 2 \sum_i b_i^* D_i u + c^* u, \quad a^*_{ij} = a^*_{ji} \]
in which the coefficients satisfy the same conditions as the coefficients in (1). \( L^* \) is the Euler-Jacobi operator associated with the quadratic functional \( J^* \) defined by
\[ J^*[u] = \int_R \left[ \sum_{i,j} a_{ij}^* D_i u D_j u - 2u \sum_i b_i^* D_i u - c^* u^2 \right] dx. \]

Define \( V[u] = J^*[u] - J[u] \), \( u \in \Omega \). Since \( u = 0 \) on \( B \), it follows from partial integration that
\[ V[u] = \int_R \left[ \sum (a_{ij}^* - a_{ij}) D_i u D_j u \right. \]
\[ + \left\{ \sum D_i(b_i^* - b_i) + c - c^* - g \right\} u^2 \right] dx. \]

**Theorem 1.** Suppose \( g \) satisfies (3). If there exists a nontrivial solution \( u \) of \( L^* u = 0 \) in \( R \) such that \( u = 0 \) on \( B \) and \( V[u] > 0 \), then every solution of \( L^* u = 0 \) vanishes at some point of \( \bar{R} \).

**Proof.** The hypothesis \( V[u] > 0 \) is equivalent to \( J[u] < J^*[u] \). Since \( u = 0 \) on \( B \), it follows from Green’s formula that \( J^*[u] = 0 \). Hence the hypothesis \( J[u] < 0 \) of the lemma is fulfilled.

**Theorem 2.** Suppose \( g \) \( \det (a_{ij}) > - \sum b_i B_i \). If there exists a nontrivial solution of \( L^* u = 0 \) in \( R \) such that \( u = 0 \) on \( B \) and \( V[u] \geq 0 \), then every solution of \( L^* u = 0 \) vanishes at some point of \( \bar{R} \).

Since \( Q \) is positive definite, the lemma is valid when the hypothesis \( J[u] < 0 \) is replaced by \( J[u] \leq 0 \). The proof of Theorem 2 is then analogous to that of Theorem 1.
In the case that equality holds in (3), that is

\[(8)\quad g = - \sum b_i B_i / \det (a_{ij}),\]

define

\[(\delta = \sum D_i (b_i - b_i) + c - c^* - g).\]

$L$ is called a "strict Sturmian majorant" of $L^*$ by Hartman and Wintner \[4\] when the following conditions hold: (i) $(a_{ij}^* - a_{ij})$ is positive semidefinite and $\delta \geq 0$ in $\overline{R}$; (ii) either $\delta > 0$ at some point or $(a_{ij}^* - a_{ij})$ is positive definite and $c^* \neq 0$ at some point. The corollary below follows immediately from Theorem 1.

**Corollary.** Suppose that $L$ is a strict Sturmian majorant of $L^*$. If there exists a solution $u$ of $L^* u = 0$ in $R$ such that $u = 0$ on $B$ and $u$ does not vanish in any open set contained in $R$, then every solution of $L v = 0$ vanishes at some point of $\overline{R}$.

If the coefficients $a_{ij}^*$ are of class $C^{2,1}(R)$ (i.e. all second derivatives exist and are Lipschitzian), the hypothesis that $u$ does not vanish in any open set of $R$ can be replaced by the condition that $u$ does not vanish identically in $R$ because of Aronszajn's unique continuation theorem \[1\].

In the case $n = 2$ considered by Protter \[6\], the condition $\delta \geq 0$ reduces to

\[(9)\quad (a_{11}^2 - a_{12}^2) \left( \sum_{i=1}^2 D_i (b_i^* - b_i) + c - c^* \right) \geq a_{11} b_2^2 - 2 a_{12} b_1 b_2 + a_{22} b_1^2,\]

which is considerably simpler than Protter's condition. It reduces to Protter's condition

\[\sum_{i=1}^2 D_i b_i^* + c - c^* \geq 0\]

in the case that $b_1 = b_2 = 0$, and also in the case that $a_{12} = a_{12}^* = 0$, $a_{11} = a_{11}^*$, $a_{22} = a_{22}^*$. (Two incorrect signs appear in \[6\]).

The following example in the case $n = 2$ illustrates that Theorem 1 is more general than the pointwise condition (9). Let $R$ be the square $0 < x^1, x^2 < \pi$. Let $L^*$, $L$ be the elliptic operators defined by
\[ L^*u = D_1^2u + D_2^2u + 2u, \]
\[ Lv = D_1^2v + D_2^2v + D_1v + cv, \]

where
\[ c(x^1, x^2) = f(x^1)f(x^2) + 5/4, \]
and \( f \in C[0, \pi] \) is not identically zero. The function \( u = \sin x^1 \sin x^2 \) is zero on \( B \) and satisfies \( L^*u = 0 \). The condition \( V[u] > 0 \) of Theorem 1 reduces to
\[ \int_0^\pi \int_0^\pi f(x^1)f(x^2) \sin^2 x^1 \sin^2 x^2 \, dx^1 \, dx^2 > 0. \]

Since this is fulfilled, every solution of \( Lv = 0 \) vanishes at some point of \( \bar{R} \). This cannot be concluded from (9) or from Protter's result [6] unless \( f \) has constant sign in \( R \).

In the case \( n = 1 \), \( L \) is an ordinary differential operator of the form
\[ Lu = (au')' + 2bu' + cu, \]
and \( R \) is an interval \((x_1, x_2)\). We assert that \( \bar{R} \) can be replaced by \( R \) in the lemma and theorems; for \( v \) can have at most a simple zero at the boundary points \( x_1 \) and \( x_2 \), and hence the first integral on the right side of (5) exists and is nonnegative provided only that \( v \neq 0 \) in \( R \).

**Theorem 3.** If there exists a nontrivial solution \( u \) of \( L^*u = 0 \) in \( R \) such that \( u = 0 \) on \( B \) and
\[ \int_{x_1}^{x_2} \left[ (a^* - a)u'^2 + \left( b^* - b + c - c^* - \frac{b^2}{a} \right) u^2 \right] \, dx > 0, \]
then every solution of \( Lv = 0 \) has a zero in \((x_1, x_2)\).

In the self-adjoint case \( b = b^* = 0 \) it was shown by Clark and the author [2] that the strict inequality in the hypothesis \( V[u] > 0 \) of Theorem 1, and therefore also in (10), can be replaced by \( \geq \). Indeed, this is transparent when the proof of the above lemma is specialized to the self-adjoint case. With \( > \) replaced by \( \geq \), (10) reduces to Leighton's condition in the self-adjoint case [5].

**References**


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**q-ANALOGUES OF CAUCHY'S FORMULAS**

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1. Let $q$ be a given number and let $\alpha$ be real or complex. The $\alpha$th "basic number" is defined by means of $[\alpha] = (1 - q^\alpha)/(1 - q)$. This has served as a basis for an extensive amount of literature in mathematics under such titles as Heine, basic, or $q$-series and functions. The basic numbers also occur naturally in many theta identities. The works of Jackson (for bibliography see [2]) and Hahn [3] have stimulated much interest in this field.

One important operation that is intimately connected with basic series as well as with difference and other functional equations is the $q$-derivative of a function $f$. This is defined by

$$Df(x) = \frac{f(qx) - f(x)}{x(q - 1)}.$$  

Jackson defined the operations, which he called $q$-integration,

$$\int_0^x f(t)d(q, t) = x(1 - q) \sum_{k=0}^{\infty} q^k f(xq^k)$$  

and

$$\int_x^{\infty} f(t)d(q, t) = x(1 - q) \sum_{k=0}^{\infty} q^{-k} f(xq^{-k})$$

Received by the editors October 28, 1965 and, in revised form, November 30, 1965.