ON SPLITTING FIELDS FOR CERTAIN LIE ALGEBRAS OF PRIME CHARACTERISTIC

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1. Let $F$ be a field of prime characteristic different from 2 or 3 and $L$ a Lie algebra over $F$ with an abelian Cartan subalgebra $H$. For $\alpha$ in $H^*$ (the dual space of $H$) set $L_{\alpha} = \{ x \in L | [xh] = \alpha(h)x, \text{ for all } h \text{ in } H \}$, and as usual, if $L_{\alpha} \neq (0)$, $\alpha$ is called a root with respect to $H$ and $L_{\alpha}$ the root space for $\alpha$. We have $L_0 = H$ and $[L_{\alpha}L_{\beta}] \subseteq L_{\alpha + \beta}$. Seligman and Mills in [1] have called $L$ a Lie algebra of classical type if $L$ contains an abelian Cartan subalgebra $H$ and if $H$ and $L$ satisfy:

(i) $[LL] = L$.
(ii) $L$ has center $(0)$.
(iii) $L$ is a direct sum of subspaces $L_{\alpha}$.
(iv) If $\alpha$ is a nonzero root, then $[L_{\alpha}L_{-\alpha}]$ is one-dimensional.
(v) If $\alpha$ is a nonzero root and $\beta \in H^*$, then there is a positive integer $m$ such that $\beta + ma$ is not a root.

Let $L$ be a Lie algebra over $F$ such that $L_K$ is of classical type, where $K$ is the algebraic closure of $F$. An extension field $P$ of $F$ is called a splitting field for $L$ provided $L_P$ is of classical type. We can now state the main theorem of this paper as:

**Theorem 1.1.** Every semisimple Lie algebra over $F$ with nondegenerate Killing form $(x, y) = \text{tr}(\text{ad}x)(\text{ad}y)$ has a separable splitting field.

Note that if $F$ is finite and $L$ has nondegenerate Killing form then every finite extension is separable, in particular one that splits $L$. Therefore, we may assume, in what follows, that $F$ is infinite.

2. **Lemma 2.1.** If $L$ is semisimple with nondegenerate Killing form, then $L$ contains a regular element $x$ such that the minimum polynomial of $\text{ad}(x)$ has the form:

$$
\mu(\lambda) = \lambda \prod_{\alpha} (\lambda - \alpha(x))
$$

where the $\alpha(x)$ are distinct and different from zero in some extension $P$ of $F$.

**Proof.** Recall that an element $x$ in $L$ is regular provided the 0-space of $\text{ad}(x)$ has minimal dimension. If $x$ is regular in $L$ and $P$ is

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an extension of $F$ then $x$ is regular in $L_P$. To see this, let $(u_1, u_2, \ldots, u_n)$ be a basis for $L$ and $(X_1, X_2, \ldots, X_n)$ be algebraically independent indeterminants. Let $F'=F(X_1, \ldots, X_n)$. Then $X = \sum X_i u_i$ is in $L_{P'}$, and the characteristic polynomial of $\text{ad}(X)$ is given by:

$$\det(\lambda I - \text{ad}(X)) = \sum_{i=0}^{n} M_i(X) \lambda^i,$$

where $M_i(X) \in F[X_1, \ldots, X_n]$. If $x = \sum_{i=1}^{n} \xi_i u_i$, then $M_i(\xi) = 0$ for $i < r$, where $r$ is the dimension of the zero space of $\text{ad}(x)$, and $M_r(\xi) \neq 0$. Also, since $x$ is regular, $M_i(\eta_1, \eta_2, \ldots, \eta_n) = 0$ for $i < r$ and all $\eta_i$ in $F$. Thus, since $F$ is infinite, $M_i(X)$ is zero for $0 \leq i < r$ and

$$\det(\lambda I - \text{ad}(X)) = \sum_{r} M_i(X) \lambda^i.$$

Now let $y = \sum \mu_i u_j$, $\mu_j$ in $P$, i.e. $y$ in $L_P$. Then $M_i(\mu) = 0$ for $0 \leq i < r$, so that the zero space of $\text{ad}(y)$ in $L_P$ has dimension greater than or equal to $r$. Therefore $x$ is regular in $L_P$ as claimed. As in the above, for generic element $X$ in $L_{P'}$, we have:

$$\det(\lambda I - \text{ad}(X)) = \lambda^r (M_r(X) + M_{r+1}(X) \lambda + \cdots + M_n(X) \lambda^{n-r}),$$

where $M_r(X) \neq 0$. Consider $g(X, \lambda)$, where:

$$g(X, \lambda) = M_r(X) + M_r(X) \lambda + \cdots + M_n(X) \lambda^{n-r}.$$

Then $g(X, \lambda) \in F[X_1, \ldots, X_n, \lambda] \subset F(X_1, \ldots, X_n)[\lambda] = F'[\lambda]$, and thus $g$ has a discriminant given by:

$$D(X_1, \ldots, X_n) = \left( \prod_{i<j} (\rho_i - \rho_j) \right)^2,$$

where $\rho_1, \rho_2, \ldots, \rho_{n-r}$ are all roots of $g(X, \lambda)$ as a polynomial in $\lambda$ in some splitting field over $F'$, multiple ones taken as many times as their multiplicity. Now, $D(X_1, \ldots, X_n)$ is a symmetric function of the roots and therefore is in the ring generated by the elementary functions of the roots, i.e. the ring generated by $M_r(X), M_{r+1}(X), \ldots, M_{r-r}(X)$. Thus, there exists a polynomial $Q(y_r, y_{r+1}, \ldots, y_n)$ with integral coefficients such that

$$Q(M_r(X), \ldots, M_n(X)) = 0$$

if and only if $g(X, \lambda)$ has repeated roots in its splitting field. Consider $g(X) = M_r(X) Q(M_r(X), \ldots, M_n(X))$. Then $g(X) \in F[X_1, \ldots, X_n]$ and if $g(X) \neq 0$ there exist elements $\xi_1, \ldots, \xi_n$ in $F$ such that $g(\xi_1, \ldots, \xi_n) \neq 0$. Suppose such an $n$-tuple exists and set $x = \sum \xi_i u_i$, \ldots
Then $\text{ad}(x)$ has characteristic polynomial

$$
\det(\lambda I - \text{ad}(x)) = \lambda^r (M_r(\xi) + M_{r+1}(\xi)\lambda + \cdots + \lambda^{n-r}).
$$

$M_r(\xi) \neq 0$ and $M_r(\xi) + M_{r+1}(\xi)\lambda + \cdots + M_n(\xi)\lambda^{n-r}$ has distinct roots in a splitting field. Thus $x$ is regular and the minimum polynomial of $\text{ad}(x)$ has the form:

$$
\mu(\lambda) = \lambda \prod_{\alpha} (\lambda - \alpha(x)),
$$

where $\alpha(x)$ are distinct and different from zero. It remains to show that $g(X) \neq 0$. For this, let $H$ be a standard Cartan subalgebra in $L_K$ and $h_0 \in H$ be such that $\alpha(h_0)$ are distinct and nonzero for all roots $\alpha$ relative to $H$. If $h_0 = \sum \omega_iu_i$, $\omega_i \in K$, then $g(\omega_1, \cdots, \omega_n)$ is not zero, so that $g(X_1, \cdots, X_n) \neq 0$, as desired.

3. Proof of Theorem 1.1. Let $L$ be a semisimple Lie algebra over $F$ of dimension $n$, with nondegenerate Killing form, and $x$ a regular element, where the dimension of the zero-space of $\text{ad}(x)$ is $r$ and where the minimum polynomial of $\text{ad}(x)$ has the form:

$$
\mu_x(\lambda) = \lambda \prod_{\alpha} (\lambda - \alpha(x)), \quad \alpha(x) \in F,
$$

with all $\alpha(x)$ distinct, different from zero, and $n - r$ in number. We will show that $L$ is of classical type. Note that (ii) is satisfied by our hypotheses. Let $H$ be the zero space of $\text{ad}(x)$. Then $H$ is the Cartan subalgebra of $L$ which will play the role of satisfying the remaining axioms, and $H$ has dimension $r$. Since all $\alpha(x)$ are distinct and characteristic roots of $\text{ad}(x)$, the subspaces $L_{\alpha(x)}$ corresponding to $\alpha(x)$ have dimension one. Then we have

$$
L = H + \sum_{\alpha} L_{\alpha}.
$$

Now for $h \in H$, $[hx] = 0$ and if $y \in L_{\alpha}$, $[[yh]x] = [[yx]h] = \alpha(x)[yh]$. Thus $[yh] \in L_{\alpha}$, i.e. $[L_{\alpha}H] \subseteq L_{\alpha}$ for $\alpha(x) \neq 0$. Since $L_{\alpha}$ is one dimensional this means that for $e_{\alpha} \in L_{\alpha}$ and $h \in H$, $[e_{\alpha}h] = \lambda(h)e_{\alpha}$. Set $\alpha(h) = \lambda(h)$. Thus the characteristic roots of $\text{ad}(h)$ are in the ground field and $L_{\alpha}$ is a root space relative to $H$. Furthermore, the restriction to $H$ of the Killing form on $L$ is nondegenerate. To see this, let $h \in H$, $e_{\alpha} \in L_{\alpha}$, for $\alpha \neq 0$. Then $[e_{\alpha}h] = \alpha(h)e_{\alpha}$ and we can choose $e_{\alpha}^{(1)} \in L_{\alpha}$ such that $[e_{\alpha}^{(1)}h] = e_{\alpha}$. Then $(h, e_{\alpha}) = (h, [e_{\alpha}^{(1)}h]) = ([hk], e_{\alpha}^{(1)}) = 0$. Thus $(H, L_{\alpha}) = 0$ for all $\alpha \neq 0$ and the form must be nondegenerate on $H$. It follows that $H$ is abelian. In the case where $F$ is algebraically closed
this is a result due to Zassenhaus. In the general case a field extension argument gives the result. This, together with (2) now shows that axiom (iii) holds.

Thus we have that \( L \) contains an abelian Cartan subalgebra \( H \), and that relative to a fixed basis for \( L \), \( \text{ad}(h) \) has a diagonal matrix for every \( h \in H \). Next we note that if \( \alpha \) is a root different from zero, then so is \( -\alpha \). For, let \( e_\alpha \in L_\alpha, e_\beta \in L_\beta \). Then, for some \( h \in H, e_\alpha = [e_\alpha h] \), and \( (e_\beta, e_\alpha) = (e_\beta, [e_\alpha h]) = ([e_\beta e_\alpha], h) = 0 \), unless \( \beta = -\alpha \), since \([e_\beta e_\alpha] \in L_{\alpha+\beta} \). If \( L_{-\alpha} = (0) \) we have \((L, e_\alpha) = 0\), a contradiction.

The nondegeneracy of the form \((x, y)\) on \( H \) implies that for each \( \alpha \in H^* \) there exists an \( h_\alpha \in H \) such that \((h_\alpha, h) = \alpha(h)\), for all \( h \in H \).

In what follows we shall have occasion to refer to results in Seligman's Memoir [2] which we shall denote by \( M \).

**Lemma 3.1** (\( M, \text{Corollary 3.2} \)). If \( e_\alpha \in L_\alpha, e_-\alpha \in L_-\alpha \), then \([e_-\alpha e_\alpha]\) = \((e_-\alpha, e_\alpha)h\).

Since \( L \) has nondegenerate form, every derivation of \( L \) is inner [3]. Thus, for \( x \in L, \text{ad}(x)p \) is a derivation in \( L \) and there exists a unique \( y \in L \) such that \( \text{ad}(x)p = \text{ad}(y) \). Setting \( x_p = y \), \( L \) becomes a restricted Lie algebra over \( F \) and the adjoint mapping is a restricted representation of \( L \).

We turn now to a modification of two results due to Jacobson dealing with low-dimensional Lie algebras and their representations (\( M, \text{Lemma 4.1 and 4.2} \)). The modification involves replacing algebraic closure of the ground field with the fact that for the representation \( U \) we have \( U(h) \) is diagonalizable for all \( h \in H \).

**Lemma 3.2.** Let \( L \) be a two-dimensional Lie algebra over \( F \) with basis elements \( e, h, \) and \([eh] = e \). Let \( U \) be an irreducible representation of \( L \) such that \( U(h) \) and \( U(e)p \) are diagonalizable. Then either \( U(e)p = 0 \) or \( U \) is equivalent to the \( p \)-dimensional representation \( W \):

\[
W(e) = \begin{pmatrix}
0 & 1 & 0 & \cdots & 0 \\
0 & 0 & 1 & \cdots & 0 \\
0 & 0 & \cdots & 0 & 1 \\
0 & 0 & \cdots & 0 & 0 \\
0 & 0 & \cdots & 0 & 0
\end{pmatrix}, \quad W(h) = \begin{pmatrix}
\lambda & 0 \\
\lambda + 1 & \ddots \\
& & \ddots & 0 \\
& & & \lambda + p - 1
\end{pmatrix}.
\]

**Proof.** Suppose \( U(e)p = U(e)p \neq 0 \). Then, since \( U(e)p \) and \( U(h)p \) - \( U(h) \) are diagonalizable by our assumptions, these matrices are scalar. For, if \( V_1 = \{ v \mid vU(e)p = \lambda v \} \), and \( V_2 = \{ v \mid v(U(h)p - U(h)) = \mu v \} \), then these are invariant subspaces of the representation space \( V \) and since one of them is not zero for some \( \lambda \) by diagonalizability one must be the whole space. Thus in each case, \( U(e)p = \sigma I, \sigma \in F \),
and \( U(h)^p - U(h) = \rho I, \rho \subseteq F \). Now, let \( \lambda \) be a characteristic root of \( U(h) \) and \( v \neq 0 \) such that \( vU(h) = \lambda v \). Then \( vU(e)^p = \sigma v \), and the space spanned by \( \{ v, vU(e), \ldots, vU(e)^{p-1} \} \) is an invariant subspace of dimension \( p \), thus the whole space \( V \). To see the invariance, we note:

\[
(3) \quad vU(e)^kU(h) = (\lambda + k)vU(e)^k, \quad 0 \leq k \leq p - 1.
\]

Now, relative to this basis for \( V \), the matrices of \( U(h) \) and \( U(e) \) have the form of the lemma.

**Lemma 3.3.** Let \( L \) be a three-dimensional Lie algebra over \( F \) with basis \( e, f, h \) and let \([ef] = h, [fh] = 0 = [eh]\). Let \( U \) be a nonzero irreducible representation of \( L \) such that \( U(e)^p = 0 \) and \( U(f)^p = 0 \) and \( U(h) \) is diagonalizable. Then \( \text{tr}(u(e)U(f)) = 0 \).

**Proof.** Since \( U(h) \) is diagonalizable and centralizes the representation we have \( U(h) = \lambda I \). If \( \lambda = 0 \), then \( U(e)U(f) = U(f)U(e) \). Since both \( U(e) \) and \( U(f) \) are nilpotent, so is \( U(e)U(f) \), and thus \( \text{tr}(U(e)U(f)) = 0 \). Suppose now that \( \lambda \neq 0 \) and let \( v \neq 0 \) be an element of the representation space \( V \) such that \( vU(f) = 0 \). Such a \( v \) exists since \( U(f) \) is nilpotent. Now, let \( K \) be the space spanned by \( \{ v, vU(e), \ldots, vU(e)^{p-1} \} \). Then \( KU(e) \subseteq K \) and \( KU(h) \subseteq K \). Furthermore, we have:

\[
(4) \quad vU(e)^kU(f) = vU(e)^kU(f)U(e) + vU(e)^{k-1}U(h).
\]

Actually, by induction we have:

\[
(5) \quad vU(e)^kU(f) = k\lambda(vU(e)^{k-1}), \quad k \geq 1.
\]

Thus, \( K = V \), the whole space, and thus \( \{ v, vU(e), \ldots, vU(e)^{p-1} \} \) is a basis for \( V \). The matrices relative to this basis are:

\[
U(e) = \begin{bmatrix}
0 & 1 & 0 & \cdots & 0 \\
0 & 0 & 1 & \cdots & 0 \\
0 & 0 & 0 & \cdots & 1 \\
0 & 0 & 0 & \cdots & 0
\end{bmatrix}, \quad U(f) = \begin{bmatrix}
0 & 0 & 0 & \cdots & 0 \\
\lambda & 0 & 0 & \cdots & 0 \\
0 & 2\lambda & \cdots & 0 \\
0 & 0 & \cdots & (p-1)\lambda & 0
\end{bmatrix}
\]

and thus \( \text{tr}(U(e)U(f)) = 0 \) as claimed.

Using these lemmas together with our diagonalizability condition we can now prove the following analogues of the required theorems in \( M \).

**Theorem 3.1 (M, Theorem 4.1).** If \( \alpha \neq 0 \) is a root then \( e^\alpha_\alpha = 0 \).

**Proof.** For \( h \in H, [he^\alpha_\alpha] = 0 \), thus \( e^\alpha_\alpha \subseteq H \). Choose \( h \) such that \( \alpha(h) = 1 \). Then \( \{ e_\alpha, h \} \) forms a two-dimensional Lie algebra \( L_1 \) as in Lemma 3.2. For the representation \( U(x) \) take \( \text{ad}_L(x) \). Then, the restriction of \( U \) to \( L_1 \) can be written in the form:
Furthermore, since \( \text{ad}_L(h) \) are diagonal for \( h \in H \), the same holds for \( \text{ad}_L(h) \) restricted to \( M \), \( M \) an irreducible \( L_1 \) submodule of \( L \), and for the transformations induced by \( \text{ad}_L(h) \) in \( L/M \). Continuing this argument on \( L/M \) we see that \( U_i(h) \) is a diagonal matrix relative to a suitable basis for each \( h \in H \).

Now \( (ea, h) = \text{tr}(U(ea)^p U(h)) \) and either \( U_i(ea)^p = 0 \) or \( U_i(ea) \) has the form of Lemma 3.2. In any case, \( \text{tr}(U_i(ea)^p U_i(h)) \) is zero so that \( (ea, h) = 0 \). This holds whenever \( \alpha(h) \neq 0 \). If \( \alpha(h) = 0 \), let \( h_1 \in H \) be chosen such that \( \alpha(h_1) \neq 0 \). Then \( \alpha(h+h_1) \neq 0 \) and \( (ea, h_1) = (e_a^p, h_1) = 0 \). Thus, \( (ea, H) = 0 \), which gives \( e_a^p = 0 \).

**Theorem 3.2 (M, Theorem 4.2).** If \( \alpha \neq 0 \) is a root, then \( \alpha(h_a) \neq 0 \).

**Proof.** Suppose \( \alpha(h_a) = 0 \) and \( e_a \neq 0 \), \( e_a \in L_\alpha, e_{-a} \neq 0, e_{-a} \in L_{-a} \) such that \( (e_a, e_{-a}) = 1 \). By Lemma 3.1 \([e_{-a}e_a] = h_a \). Let \( L_1 \) be the algebra spanned by \( \{e_a, e_{-a}, h_a\} \). For the irreducible constituent \( U_i \) of the restriction of \( U = \text{ad}_L \) to \( L_1 \) we have \( U_i(e_a)^p = 0 \) and \( U_i(e_{-a})^p = 0 \). Thus by Lemma 3.3 we obtain \( \text{tr}(U_i(e_{-a}) U_i(e_a)) = 0 \). Therefore, \( \text{tr}(U(e_{-a}) U(e_a)) = 0 \), i.e. \( (e_{-a}, e_a) = 0 \), a contradiction.

Now, by Lemma 3.1 and Theorem 3.2 we have \([L_\alpha L_{-\alpha}]\) is one dimensional, giving us axiom (iv). Theorems 3.1 and 3.2 make it possible now to use the results of §5 of M. (For a complete proof of M, Lemma 5.1 see [1].) In particular, axiom (v) for our algebras is a consequence of Theorems 5.2 and 5.4 in M.

Finally, for axiom (i), we note that \([LL]_K = [L_K L_K] = L_K\), \( K \) the algebraic closure of \( F \). Hence we have:

\[
\dim_F [LL] = \dim_K [LL]_K = \dim_K L_K = \dim_F L.
\]

Therefore, \([LL] = L\). Thus, our algebra \( L \) is of classical type and this together with Lemma 2.1 proves Theorem 1.1.

**Bibliography**


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