# PROJECTIONS AND RETRACTIONS

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The main result of this paper is a generalization of a theorem of R. S. Phillips [6] on the nonexistence of projections of $l^\infty$ onto $c_0$. If $S$ is a locally compact Hausdorff space then let $C(S)$ denote the space of bounded continuous real (or complex) valued functions on $S$; also, let $C_0(S)$ be those functions in $C(S)$ which vanish at infinity. If $N$ is the space of positive integers with the discrete topology then $l^\infty = C(N)$ and $c_0 = C_0(N)$. Thus, Phillips' theorem says that in the case where $S = N$ there is no bounded projection of $C(S)$ onto $C_0(S)$; that is, the space $N$ does not have the *projection property.*

It is natural to ask for a characterization of spaces with the projection property in terms of their topology. Unfortunately, we cannot achieve this but we do show in Theorem 2 that pseudocompactness is a necessary condition for this property ($S$ is *pseudocompact* if and only if every real valued continuous function on $S$ is bounded). However, as Example 3 demonstrates, pseudocompactness is not sufficient.

As a corollary to Theorem 2 we obtain a result of W. W. Comfort [1]. Namely, if $S$ is completely regular and there is a retraction of its Stone-Čech compactification $\beta S$ onto $\beta S \backslash S$ then $S$ is a locally compact peudocompact space. To establish his result Comfort appealed to a result of W. Rudin which depends on the continuum hypothesis. Hence, not only do we give a relatively simple proof of Comfort's theorem, but furthermore, we do so without using the continuum hypothesis.

Before proceeding to the main theorem we will need the following theorem of I. Glicksberg [3]. The proof is not difficult and we shall not repeat it here.

THEOREM 1. *A completely regular Hausdorff space $S$ is pseudocompact if and only if for every sequence $\{V_n\}$ of nonvoid open sets with pairwise disjoint closures there is an $s_0 \in S$ such that for every integer $n_0$ and every open neighborhood $V_0$ of $s_0$ there is an $n > n_0$ with $V_0 \cap V_n^- \neq \square$.*

We can now prove our main theorem.

THEOREM 2. *Let $S$ be a locally compact Hausdorff space; if there is a bounded projection of $C(S)$ onto $C_0(S)$ then $S$ is pseudocompact.*

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Proof. Let $P$: $C(S)\to C_0(S)$ be the hypothesized projection and suppose $S$ is not pseudocompact. By Theorem 1 we can find a sequence $\{V_n\}$ of nonvoid open sets with pairwise disjoint closures such that $\{V_n^-\}$ has no cluster points; i.e., $\bigcup_{n=1}^{\infty} V_n^-$ is closed.

For every integer $n\geqq 1$ let $s_n\in V_n$ and choose a neighborhood $U_n$ of $s_n$ such that $U_n^-$ is compact and contained in $V_n$. Using Urysohn's lemma we may find a function $\phi_n$ in $C(S)$ with $0\leqq\phi_n\leqq 1$, $\phi_n(s_n)=1$, and $\phi_n(s)=0$ for $s\notin U_n$. Let $F=\bigcup_{n+1}^{\infty} U_n^-$; then $s\in F^-\subset\bigcup_{n=1}^{\infty} V_n^-$ implies that $s\in V_n^-$ for a unique $n$. It easily follows that $s\in U_n^-$, and so $F$ is closed.

If $\xi=\{x^{(n)}\}_{n=1}^{\infty}\in l^\infty$ then we define

$$f_\xi(s) = \sum_{n=1}^{\infty} x^{(n)}\phi_n(s)$$

for all $s\in S$. Since the sets $\{V_n^-\}$ are pairwise disjoint and each $\phi_n$ vanishes off $V_n^-$, we have that at most one term in the sum is not zero and so $f_\xi$ is well defined. For this same reason and because each $\phi_n$ achieves its supremum, $\|f_\xi\|_\infty=\|\xi\|_\infty$. To see that $f_\xi$ is continuous let $\{s_i\}$ be a net in $S$ which converges to a point $s\in S$. If $s\notin F$ then $\{s_i\}$ is eventually outside of $F$ because $F$ is closed. Hence, there is an $i_0$ such that for $i\geqq i_0$, $f_\xi(s_i)=f_\xi(s)=0$ and so $\{f_\xi(s_i)\}$ converges to $f_\xi(s)$. If $s\in F$ then $s\in U_n^-\subset V_n$ for a unique integer $n$. Thus, there is an $i_0$ such that for $i\geqq i_0$, $s_i\in V_n$. But $s_i\in V_n$ implies $f_\xi(s_i)=x^{(n)}\phi_n(s_i)$ and so $\{f_\xi(s_i)\}$ converges to $f_\xi(s)$. Therefore $f_\xi\in C(S)$.

Define the linear transformation $T_1$: $l^\infty\to C(S)$ by $T_1\xi=f_\xi$; then $T_1$ is an isometry. Furthermore, if $\xi\in c_0$ then $T_1\xi\in C_0(S)$.

If $\phi\in C_0(S)$ and $\epsilon>0$ then $K=\{s: |\phi(s)|\geqq\epsilon\}$ is compact, and so $K\cap U_d^-\neq\square$ for at most a finite number of $U_n^-$. Let $n_0$ be such that for $n\geqq n_0$, $K\cap U_n^-=\square$. Thus, if $n\geqq n_0$, $|\phi(s_n)|<\epsilon$; i.e., $\{\phi(s_n)\}_{n=1}^{\infty}$ is in $c_0$. Define $T_2$: $C_0(S)\to c_0$ by $T_2\phi=\{\phi(s_n)\}$; then $T_2$ is linear and bounded.

From these facts we have that $\pi$: $l^\infty\to c_0$ defined by $\pi=T_2\circ P\circ T_1$ is a bounded linear transformation. But $\xi\in c_0$ implies $T_1\xi\in C_0(S)$ and, since $P$ is a projection onto $C_0(S)$, we have $P(T_1\xi)=T_1\xi$. Moreover, since $\phi_n(s_n)=1$ for all $n$, we have that $\pi\xi=T_2(T_1\xi)=\xi$ and $\pi$ is a projection of $l^\infty$ onto $c_0$. This contradicts the result of Phillips and so $S$ must be pseudocompact. This completes the proof.

To prove Comfort's theorem we will need the following easy lemma.

Lemma. *Let $X$ be a compact Hausdorff space and $S$ an open subset of $X$; if $f\in C(X)$ then let $f_S$ be the restriction of $f$ to $S$. Then the mapping which takes each $f$ into $f_S$ defines an isometric isomorphism between*

*$C_0(S)$ and the subspace of $C(X)$ consisting of all functions vanishing on $X\backslash S$.*

COROLLARY (COMFORT). *If $S$ is a completely regular Hausdorff space and there is a retraction $r$ of $\beta S$ onto $\beta S\backslash S$ then $S$ is locally compact pseudocompact.*

PROOF. Since $\beta S\backslash S=r(\beta S)$ we have that $S$ is open in $\beta S$ and hence locally compact. Define $P_1$: $C(\beta S)\rightarrow C(\beta S)$ by $P_1^2f=f\circ r$; then $P_1$ is a bounded linear transformation. Also $P_1^2f=P_1(f\circ r)=(f\circ r)\circ r=f\circ r=P_1f$ so that $P_1$ is a projection. Finally, $P_1f=0$ if and only if $f(\beta S\backslash S)=0$. By the lemma we have that $P_1f=0$ if and only if the restriction of $f$ to $S$ belongs to $C_0(S)$. Hence, if $i$: $C(S)\rightarrow C(\beta S)$ is the map which takes each function in $C(S)$ onto its unique extension to $\beta S$, then $P=i^{-1}\circ P_1\circ i$ is a projection on $C(S)$ with $C_0(S)$ as kernel. Therefore $I-P$ is a projection of $C(S)$ onto $C_0(S)$ and, by Theorem 2, $S$ must be pseudocompact. This completes the proof.

We say that $S$ has the *retraction property* if and only if there is a retraction of $\beta S$ onto $\beta S\backslash S$. Thus, by the preceding corollary, the retraction property implies the projection property, which in turn implies pseudocompactness. That the reverse implications are false will be illustrated by some examples. Moreover, the projection property is preserved by taking finite Cartesian products while the retraction property is not [**1**].

THEOREM 3. *If $S_1$ and $S_2$ are locally compact spaces which have the projection property then so does $S_1\times S_2$.*

PROOF. By Theorem 2, $S_1$ and $S_2$ are pseudocompact; since they are also locally compact, it follows from a result of Glicksberg [**4**] that $S_1\times S_2$ is pseudocompact and $\beta(S_1\times S_2)=\beta S_1\times\beta S_2$. If $P_i$: $C(\beta S_i)\rightarrow C(\beta S_i)$ is a projection with image $C_0(S_i)$, $i=1$, 2 (for notational reasons we will identify $C_0(S_i)$ with the space of functions in $C(\beta S_i)$ which vanish on $\beta S_i\backslash S_i$), then define

$$P_1\otimes P_2\colon C(\beta S_1)\otimes C(\beta S_2)\rightarrow C(\beta S_1\times\beta S_2)$$

by

$$P_1\otimes P_2\left(\sum_{i=1}^{n}f_i\otimes g_i\right)=\sum_{i=1}^{n}(P_1f_i)(P_2g_i)$$

where $f_i\in C(\beta S_1)$, $g_i\in C(\beta S_2)$ for $1\leqq i\leqq n$. Then $P_1\otimes P_2$ is a well defined linear map; furthermore $P_1\otimes P_2$ is continuous if $C(\beta S_1)\otimes C(\beta S_2)$ has its bi-equicontinuous topology (which is here a norm topology) and $C(\beta S_1\times\beta S_2)$ has its supremum norm (see [**5**, pp. 89–93] for most of the results needed here). But $C(\beta S_1 \times \beta S_2) = C(\beta S_1) \otimes_\lambda C(\beta S_2)$, the completion of $C(\beta S_1) \otimes C(\beta S_2)$ in the bi-equicontinuous norm. Therefore $P_1 \otimes P_2$ can be extended to a bounded linear transformation

$$P: C(\beta S_1 \times \beta S_2) \to C(\beta S_1 \times \beta S_2).$$

Now, $(P_1 \otimes P_2)^2 = P_1 \otimes P_2$ implies $P$ is a projection and hence has a closed image. But the image of $P_1 \otimes P_2$ is the set of all functions of the form $\sum \phi_i \psi_i$ where $\phi_i \in C_0(S_1)$ and $\psi_i \in C_0(S_2)$. Hence, the image of $P_1 \otimes P_2$ is dense in $C_0(S_1 \times S_2)$ which implies that $C_0(S_1 \times S_2)$ is the image of $P$. Therefore $S_1 \times S_2$ has the projection property and the proof is complete.

EXAMPLES. (1) If $\Omega_0$ is the space of ordinal numbers less than the first uncountable ordinal $\Omega$ with the order topology, then $\beta\Omega_0 = \Omega_0 \cup \{\Omega\}$, the one-point compactification of $\Omega_0$ [2, p. 75]. Clearly $\Omega_0$ has the retraction property and hence the projection property. Comfort [1] showed that $\Omega_0 \times \Omega_0$ does not have the retraction property, but by Theorem 3, it does have the projection property. A more simple example of such a space is the following.

(2) Let $L_1$, $L_2$ be two long lines; i.e., $L_1$ and $L_2$ are the spaces $\Omega_0 \times [0, 1)$ with the order topology obtained from the lexicographic ordering. Let $L$ be the quotient space gotten by identifying the first points of $L_1$ and $L_2$. Then $L$ is a "long line which is long in two directions." Then $\beta L = L \cup \{\Omega, -\Omega\}$ and let the closure of $L_1$ in $\beta L$ equal $L_1 \cup \{\Omega\} = \beta L_1$. As Comfort points out, $L$ does not have the retraction property since $\beta L$ is connected and $\beta L \backslash L$ consists of two points.

However, if $g \in C(\beta L)$ such that $0 \leqq g \leqq 1$, $g(L_2) = 0$ and $g(\Omega) = 1$ then

$$Pf = f - [f(\Omega) - f(-\Omega)]g - f(-\Omega)$$

is a bounded projection onto $C_0(L)$.

(3) We will now give an example of a pseudocompact space which does not have the projection property. Let $\Lambda = \beta R \backslash (\beta N \backslash N)$ (see $6p$ of [2]).[1] Then $\Lambda$ is pseudocompact and $\beta\Lambda = \beta R$. Let $\phi_n \in C(R)$ such that $0 \leqq \phi_n \leqq 1$, $\phi_n(n) = 1$, and $\phi_n(s) = 0$ for $|s - n| \geqq 1/3$. Then, as in the proof of Theorem 2, for every $\xi = \{x^{(n)}\}$ in $l^\infty$,

$$f_\xi = \sum_{n=1}^{\infty} x^{(n)} \qquad \phi_n \text{ is in } C(R).$$

[1] We would like to thank Dr. Stelios Negrepontis for calling our attention to this space.

Hence $f_\xi$ has an extension to $\beta R$. But $\beta R=\beta\Lambda$ implies that $T_1$: $l^\infty \to C(\Lambda)$ defined by $T_1\xi=f_\xi$ is a linear isometry. Also $T_2$: $C(\Lambda) \to l^\infty$ defined by $T_2 f=\{f(n)\}$ is a bounded linear transformation. Therefore, if there is a bounded projection $P$ of $C(\Lambda)$ onto $C_0(\Lambda)$ then $T_2 \circ P \circ T_1$ is a bounded projection of $l^\infty$ onto $c_0$, contradicting the theorem of Phillips.

Let us close by pointing out that if $S$ is a locally compact non-pseudocompact space then Theorem 2 says there is no simultaneous extension of $C(\beta S\backslash S)$ into $C(\beta S)$ (see [7] for details).

## References


**1.** W. W. Comfort, *Retractions and other continuous maps from $\beta X$ onto $\beta X\backslash X$*, Trans. Amer. Math. Soc. **114** (1965), 1–9.

**2.** L. Gillman and M. Jerison, *Rings of continuous functions*, Van Nostrand, Princeton, N. J., 1960.

**3.** I. Glicksberg, *The representation of functionals by integrals*, Duke Math. J. **19** (1952), 253–261.

**4.** ———, *Stone-Čech compactifications of products*, Trans. Amer. Math. Soc. **90** (1959), 369–382.

**5.** A. Grothendieck, *Produits tensoriels topologiques et éspaces nucléaires*, Mem. Amer. Math. Soc. No. 16, Amer. Math. Soc., Providence, R. I., 1955.

**6.** R. S. Phillips, *On linear transformations*, Trans. Amer. Math. Soc. **48** (1940), 516–541.

**7.** Z. Semadeni, *Isomorphic properties of Banach spaces of continuous functions*, Studia Math. (Seria Specjalna) **1** (1963), 93–108.


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