A METRIZATION THEOREM

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A characterization of metrizable topological spaces in terms of subtopologies is given. First, several terms are defined in order to describe the pertinent subtopologies. Then, the characterization is readily established as a result of a metrization theorem due to Bing [1] and a metrization theorem due to Ceder [2].

Definition 1. Let \((X, \mathcal{T})\) be a topological space having the property that the intersection of an arbitrary number of open sets is open. Then \((X, \mathcal{T})\) is called a fundamental topological space.

Definition 2. Let \((X, \mathcal{T})\) be a topological space with a base \(\{B_j : j \in J\}\) such that for each \(x \in \bigcap \{B_i : i \in I\}\) there exists \(B_x \in \mathcal{B}\) where \(x \in B_x \subset \bigcap \{B_i : i \in I\}\). Then, \(\mathcal{B}\) is called a fundamental base for \(\mathcal{T}\).

Remark. If \((X, \mathcal{T})\) has a fundamental base for \(\mathcal{T}\), it is a fundamental space.

Definition 3. Let \((X, \mathcal{T})\) be a topological space. Let \(\mathcal{P}_i\), a subcollection of \(\mathcal{T}\), be a base for some topology, \(\mathcal{T}_i\), on \(X\). Consider the subcollection of \(\mathcal{T}\), \(\overline{\mathcal{P}}_i\), where \(\overline{\mathcal{P}}_i = \{X - \overline{T}_i : T_i \in \mathcal{T}_i\}\) and the closure is with respect to \(\mathcal{T}\). Then, \(\overline{\mathcal{P}}_i\) is a base for a topology on \(X\), \(\mathcal{T}/\mathcal{T}_i\). The topology, \(\mathcal{T}/\mathcal{T}_i\), is called the dual topology of \(\mathcal{T}_i\) relative to \(\mathcal{T}\). \(\overline{\mathcal{P}}_i\) is called the dual base of \(\mathcal{P}_i\).

We state two metrization theorems, without proof, to be used in the establishment of a new metrization theorem. For other proofs see [1], [2].

**A Metrization Theorem of Bing.** A topological space is metrizable if and only if it is \(T_1\) and regular and the topology has a \(\sigma\)-discrete base.

**A Metrization Theorem of Ceder.** A topological space \((X, \mathcal{T})\) is metrizable if and only if it is \(T_1\) and regular and the topology has a \(\sigma\)-closure preserving base, \(\mathcal{B} = \bigcup_{i=1}^{\infty} \{B_i\}\), such that for each \(x \in X\), and \(n\), a positive integer, \(\bigcap \{B : x \in B, B \in B_n\}\) is a neighborhood of \(x\).

We now state a metrization theorem which characterizes a topological space, \((X, \mathcal{T})\), as metrizable in terms of a countable family of bases, \(\mathcal{B}_i, i = 1, 2, \ldots\) for subtopologies, \(\mathcal{T}_i\) of \(\mathcal{T}\), and in terms of the dual bases, \(\overline{\mathcal{B}}_i, i = 1, 2, \ldots\). More precisely:

**Theorem.** A topological space \((X, \mathcal{T})\) is metrizable if and only if it is \(T_1\) and regular and has a base \(\mathcal{B} = \bigcup_{i=1}^{\infty} \mathcal{B}_i\) where for each \(i, \mathcal{B}_i\) is a fun-

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**Fundamental base for a topology, \( 3_i \), on \( X \), and for each \( i \), the dual base, \( \bar{\beta}_i \), is a fundamental base for \( 3/3_i \), the dual topology of \( 3_i \), relative to \( 3 \).**

**Proof.** Let \((X, 3)\) be a topological space satisfying the hypotheses above. Then, we shall show that \((X, 3)\) satisfied the hypotheses of Ceder’s Theorem. Let \( \beta = \bigcup_{i=1}^{\infty} \beta_i \) be the given base for \( 3 \). Then, for each \( x \in X \), \( n \) a positive integer, \( \bigcap \{ B : x \in B, B \in \beta_n \} \) is a neighborhood of \( x \) in \( 3_n \subset 3 \). Also, \( X - \bigcup \{ \bar{B}_i : B_i \in \beta_n \} \subset \beta_n \) \( \in 3/3_i \subset 3 \). Therefore, \( \bigcup \{ \bar{B}_i \} \) is closed. Hence, \( \bigcup \{ \bar{B}_i \} = \text{cl} [ \bigcup B_i ] \). Thus, \( \beta_n \) is a closure preserving family.

Now, let \((X, 3)\) satisfy the hypotheses of Bing’s Theorem, and let \( \beta = \bigcup_{i=1}^{\infty} \beta_i \) be the \( \sigma \)-discrete base. We define \( \beta'_i = \beta_i \cup \{ X \} \), \( i = 1, 2, \ldots \). It is apparent that \( \beta'_i \) is a fundamental base for some subtopology, \( 3'_i \), of \( 3 \) for each \( i \). It remains only to show that \( \bar{\beta}'_i \) is a fundamental base, also. Now, \( \bigcap a \{ X - T_a : T_a \in \beta'_i \} = X - \bigcup a \{ T_a \} = X - \text{cl} [ \bigcup a T_a ] \) \( \in 3/3'_i \), since \( \beta'_i \) is a closure preserving subfamily of \( 3 \) implies that \( 3'_i \) is also a closure preserving subfamily of \( 3 \). Thus, \( \bar{\beta}'_i \) is a fundamental base.

**References**


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