A PARACOMPACT SEMI-METRIC SPACE WHICH IS NOT AN $M_3$-SPACE\textsuperscript{1}

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E. A. Michael has shown that a regular Hausdorff space $S$ is paracompact if and only if any of the following is true: (1) every open cover of $S$ has a $\sigma$-locally finite open refinement \cite{8}, (2) every open cover of $S$ has an open $\sigma$-closure preserving refinement \cite{9}, or (3) every open cover of $S$ has an open $\sigma$-cushioned refinement \cite{10}. The Nagata-Smirnov Theorem \cite{12} or \cite{14} (see also Bing’s Theorem 3 in \cite{1}) states that a $T_3$-space with a $\sigma$-locally finite base is metrizable.

In \cite{3} Jack Ceder defines an $M_1$-space to be a $T_3$-space with a $\sigma$-closure preserving base and an $M_3$-space to be a $T_1$-space with a $\sigma$-cushioned pair base (see Definition 1 below). By Michael’s theorems cited above both $M_1$- and $M_3$-spaces are paracompact, and in view of the Nagata-Smirnov Theorem, one might suspect that they would even be metrizable. In \cite{3}, however, Ceder shows that such is not the case; in fact $M_1$- and $M_3$-spaces need not be first countable, and they need not be metrizable even if they are first countable. Ceder does show, though, that a first countable $M_3$- (and hence $M_1$-) space is a paracompact semi-metric space, and he raises the question: is this a characterization—i.e. is every paracompact semi-metric space an $M_3$-space (and perhaps even an $M_1$-space)? A negative answer is given to that question in this paper.

Also it was called to my attention by Professor Michael that a question raised by C. Borges \cite{2} is answered (in the negative) by the same example below, which is a cosmic space (the regular continuous image of a separable metric space \cite{11}) that is not an $M_3$-space.

**Definition 1.** Let $P$ be a collection of ordered pairs of subsets of the $T_1$-space $S$ such that, for each $p = (p_1, p_2) \in P$, $p_1$ is open and $p_1 \subseteq p_2$, and such that, for every $x \in S$ and every neighborhood $U$ of $x$, there is a $p \in P$ for which $x \in p_1 \subseteq p_2 \subseteq U$. Then $P$ is called a pair base for $S$. Moreover, $P$ is called cushioned if, for every $Q \subseteq P$,

$$\text{Cl} \cup \{p_1: p \in Q\} \subseteq \cup \{p_2: p \in Q\}$$

and $P$ is $\sigma$-cushioned if it is the union of countably many cushioned collections.

An $M_3$-space is a $T_1$-space with a $\sigma$-cushioned pair base.

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Note that Borges calls $M_3$-spaces stratifiable spaces in [2]. Also, first countable $M_3$-spaces are sometimes called Nagata spaces [3].

**Definition 2.** A $T_1$-space $S$ is a semi-metric space provided that there is a function $d$ from $S \times S$ into the nonnegative reals such that

1. for every $(x, y) \in S \times S$, $d(x, y) = d(y, x)$ and $d(x, y) = 0$ if and only if $x = y$ and
2. for every $x \in S$ and $M \subseteq S$, $\inf \{d(x, y) : y \in M\} = 0$ if and only if $x \in \text{Cl} M$ (i.e., "the topology is invariant with respect to $d$").

All other terms are defined as in [7] or [13].

Alternative characterization of semi-metric spaces (Theorem 3.2 of [4]): A $T_1$-space $S$ is semi-metric if and only if there is a collection $\{g(n, x) : x \in S, n = 1, 2, \ldots\}$ of open sets such that (1) for each $x \in S$, $\{g(n, x) : n = 1, 2, \ldots\}$ is a local base for the topology at $x$ and (2) if $y \in S$ and $x$ is a point sequence such that, for each $m$, $y \in g(m, x_m)$, then $x$ converges to $y$.

**Theorem 1.** There exists a regular Lindelöf (and hence paracompact) semi-metric space $S$ which is a cosmic space (the continuous image of a separable metric space) but is not an $M_3$-space.

**Proof.** Such a space $S$ is defined as follows. The points of $S$ are those points $z$ of the complex plane such that either (1) $\text{Im} z = 0$ and $\text{Re} z$ is irrational or (2) $\text{Im} z > 0$ and both $\text{Im} z$ and $\text{Re} z$ are rational. For each point $z \in S$ of type 1 (i.e., $\text{Im} z = 0$) and each natural number $n$, $B(n, z) = \{x \in S : \text{Im} x < |\text{Re}(x-z)| < 1/n$ or $z = x\}$ (i.e. the "bow-tie region" of radius $1/n$ and vertex angle $45^\circ$) is a basis element for $S$; and for each $z \in S$ of type 2 and for each $n$, the open disc of radius $1/n$ and center $z$ is a basis element. The space is clearly regular since, for every $z \in S$ with $\text{Im} z = 0$ and for each $n$, $B(n, z)$ has only the two points $z+1/n$ and $z-1/n$ of $S$ on its boundary (all points of its boundary in the complex plane having at least one coordinate irrational). Also $S$ is easily seen to be semi-metric by the above alternative characterization, and $S$ is clearly Lindelöf. That $S$ is a cosmic space follows easily by the same argument given for Example 12.1 in [11]. Finally, $S$ is not an $M_3$-space. For suppose that there were a $\sigma$-cushioned pair base $P = \cup \{Q(i) : i = 1, 2, \ldots\}$ for $S$. Then there would be a second category subset $M$ of the $x$-axis, with $M \subseteq S$, and natural numbers $m$ and $k$ such that, for each $x \in M$, there is a $q \in Q(k)$ such that

$B(m, x) \subset q_1 \subset q_2 \subset B(1, x)$.

Pick a rational number $r$ in the closure (with respect to the Euclidean topology) of $M$. Then $r$ is in the closure of $\{x \in M : x > r\}$ or of
\{x \in M : x < r\}; assume the former. Let \( R \subset Q(k) \) consist of all \( q \in Q(k) \) which satisfy (*) for some \( x \in M \) with \( r < x < r + 1/m \). Then, for \( y = (r + 1/m, 1/m) \), \( y \in \text{Cl} \cup \{ p_1 : p \in R \} \) but \( y \notin \bigcup \{ p_2 : p \in R \} \). That contradicts the assumption that \( Q(k) \) is cushioned. Thus \( S \) is not an \( M_3 \)-space.

The questions remain (1) whether every \( M_3 \)-space is an \( M_1 \)-space (see [3] for the definition of an \( M_2 \)-space, an “intermediate” space), (2) what topological condition is necessary in order for a paracompact semi-metric space to be an \( M_2 \)-space, (3) whether every Lindelöf semi-metric space is a cosmic space, (4) whether every separable \( M_3 \)-space is a cosmic space (see [2]), and (5) whether every regular countable space is an \( M_3 \)-space. For some theorems relating semi-metric and \( M_3 \)-spaces see [5], and for a necessary and sufficient condition that an \( M_3 \)-space be metrizable (namely: that it have a point-countable base) see [6].

**References**


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