A NOTE ON THE BRAUER GROUP[1]

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1. **Introduction.** Let F be a field, K an extension field; let $\mathcal{B}(K/F)$ denote the Brauer group of classes of simple algebras with center F split by K. $\mathcal{B}(K/F)$ is a functor in both K and F which is "left exact" in K: if L is an extension of K, the injection $K \to L$ induces an injection $\mathcal{B}(K/F) \to \mathcal{B}(L/F)$.

Now assume that F is a field of characteristic $p>0$, and let C be a purely inseparable, K_s a separable extension of F, both finite. It is well known that $C \otimes_F K_s$ is a field, which we shall denote by K; K is the direct sum ("inverse product") of C and K_s in the category of finite extension fields of F. One might expect that since $\mathcal{B}(\ /F)$ is left exact, this property of K is reflected in $\mathcal{B}(K/F)$. Indeed, if $[K_s: F]$ and $[C: F]$ are relatively prime, one sees easily that $\mathcal{B}(K/F)$ is the direct sum of $\mathcal{B}(C/F)$ and $\mathcal{B}(K_s/F)$. However, if $[K_s: F]$ is also a pth power, $\mathcal{B}(C/F)$ and $\mathcal{B}(K_s/F)$ will in general have a nontrivial intersection; an algebra class over F can be split by both C and K_s. The question arises: is $\mathcal{B}(K/F)$ generated by its subgroups $\mathcal{B}(C/F)$ and $\mathcal{B}(K_s/F)$? The purpose of this note is to give an example which answers this question in the negative.

2. **The example.** As in the introduction, let F be a field of characteristic $p>0$, and let $C=F(\eta)$ where $\eta^p \in F$, $\eta \notin F$; assume further that $C^p=F$. Let K_s be a cyclic extension of F, $[K_s: F]=p$. Let $K=K_s \otimes C$ be the composite extension field. Finally, we assume that there exists a division algebra D with center F and maximal commutative subfield K. The construction of a specific F, C, K_s and D will be done in the 3rd section. The Brauer class of D, $[D]$, is thus an element of $\mathcal{B}(K/F)$.

THEOREM. *$[D] \in \mathcal{B}(K/F)$ is not the product of $\alpha \in \mathcal{B}(K_s/F)$ and $\beta \in \mathcal{B}(C/F)$.*

PROOF. Suppose $[D]=\alpha \cdot \beta$; we derive a contradiction. Let $A \in \alpha$ be central simple/F with maximal commutative subfield K_s, $B \in \beta$ have C as maximal commutative subfield [1, Theorem 4.27, p. 61]. We note that neither α nor β is the identity element of $\mathcal{B}(K/F)$, since D is not split by C or K_s. We have $[A: F]=p^2=[B: F]$, and since neither A nor B are matrices over F, they must both be division alge-

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bras [1, Theorem 3.18, p. 43]. By hypothesis $A \otimes B \cong D \otimes \mathfrak{M}$ where $\mathfrak{M}$ is a full matrix ring over F; however, a dimension count shows $\mathfrak{M} = F$, and $A \otimes B \cong D$. Thus we can consider A and B as division subalgebras of D, both with center F and each the centralizer of the other [1, Theorem 4.13, p. 53].

Now A is a cross-product, indeed a cyclic algebra: $A = K_s \oplus yK_s \oplus \cdots \oplus y^{p-1}K_s$, where y satisfies the following properties: $y^{-1}ky = k^\sigma$ ($\sigma \in \mathcal{G}(K_s/F)$, $\sigma \neq 1$) and $y^p \in F$ [1, p. 74]. Considered as an element of D, y must commute with B and hence with $C \subset B$. Thus $C(y)$ is a field contained in D. On the one hand, $C(y)$ must be a bigger field than C, since if y were in C it would commute with K_s; on the other hand, since $y^p \in F$ and $C^p = F$, $y \in C(y)$ must be in C. This contradiction proves the theorem.

3. **Construction of** D. We must exhibit: a field F, a division algebra D with center F and degree p^2 over F (i.e. $[D: F] = p^4$); a subfield of D, K_s, containing F and cyclic over F; a field C such that $F \subset C \subset D$ and $C^p = F$, such that C and K_s commute elementwise.

Let $F = GF(p)(\pi)$ where π is an indeterminate over $GF(p)$. Let R be the (unique) cyclic extension of $GF(p)$ of degree p^2. Set $L = R(\pi)$; L is a cyclic extension of F of degree p^2. Let σ be the generating automorphism of $\mathcal{G}(L/F)$; $\sigma^{p^2} = 1$. Finally, form $D = (L, \sigma, \pi)$ [1, p. 74].

One sees immediately via polynomial degree considerations that the indeterminate π is not a norm from $K_s = \{k \in L: k^{\sigma^p} = k\}$. This suffices to show that D is a division algebra [1, Theorem 7.19, p. 98]. Note that the field K_s is cyclic of degree p over F.

We construct C as follows: by definition of D there exists an element $y \in D$ such that $y^{p^2} = \pi$ (y induces the automorphism σ in $L \subset D = (L, \sigma, \pi)$). Let $C = F(y^p)$; then, since $(y^p)^p = \pi$, $C^p = F$. Note that C and K_s commute, since y^p and K_s do; thus $K = C \otimes K_s$ is a subfield of D.

Bibliography

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