A NOTE ON THE BRAUER GROUP\(^1\)

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1. Introduction. Let \( F \) be a field, \( K \) an extension field; let \( \mathcal{B}(K/F) \) denote the Brauer group of classes of simple algebras with center \( F \) split by \( K \). \( \mathcal{B}(K/F) \) is a functor in both \( K \) and \( F \) which is “left exact” in \( K \): if \( L \) is an extension of \( K \), the injection \( K \rightarrow L \) induces an injection \( \mathcal{B}(K/F) \rightarrow \mathcal{B}(L/F) \).

Now assume that \( F \) is a field of characteristic \( p > 0 \), and let \( C \) be a purely inseparable, \( K, \) a separable extension of \( F \), both finite. It is well known that \( C \otimes_F K \), is a field, which we shall denote by \( K \); \( K \) is the direct sum (“inverse product”) of \( C \) and \( K \), in the category of finite extension fields of \( F \). One might expect that since \( \mathcal{B}(\cdot/F) \) is left exact, this property of \( K \) is reflected in \( \mathcal{B}(K/F) \). Indeed, if \( [K_\ast : F] \) and \( [C : F] \) are relatively prime, one sees easily that \( \mathcal{B}(K/F) \) is the direct sum of \( \mathcal{B}(C/F) \) and \( \mathcal{B}(K_\ast/F) \). However, if \( [K_\ast : F] \) is also a \( p \)th power, \( \mathcal{B}(C/F) \) and \( \mathcal{B}(K_\ast/F) \) will in general have a nontrivial intersection; an algebra class over \( F \) can be split by both \( C \) and \( K_\ast \).

The question arises: is \( \mathcal{B}(K/F) \) generated by its subgroups \( \mathcal{B}(C/F) \) and \( \mathcal{B}(K_\ast/F) \)? The purpose of this note is to give an example which answers this question in the negative.

2. The example. As in the introduction, let \( F \) be a field of characteristic \( p > 0 \), and let \( C = F(\eta) \) where \( \eta^p \in F \), \( \eta \notin F \); assume further that \( C^p = F \). Let \( K_\ast \) be a cyclic extension of \( F \), \( [K_\ast : F] = p \). Let \( K = K_\ast \otimes C \) be the composite extension field. Finally, we assume that there exists a division algebra \( D \) with center \( F \) and maximal commutative subfield \( K \). The construction of a specific \( F, C, K_\ast \), and \( D \) will be done in the 3rd section. The Brauer class of \( D \), \( [D] \), is thus an element of \( \mathcal{B}(K/F) \).

**Theorem.** \( [D] \in \mathcal{B}(K/F) \) is not the product of \( \alpha \in \mathcal{B}(K_\ast/F) \) and \( \beta \in \mathcal{B}(C/F) \).

**Proof.** Suppose \( [D] = \alpha \cdot \beta \); we derive a contradiction. Let \( A \in \alpha \) be central simple over \( F \) with maximal commutative subfield \( K_\ast \), \( B \in \beta \) have \( C \) as maximal commutative subfield [1, Theorem 4.27, p. 61]. We note that neither \( \alpha \) nor \( \beta \) is the identity element of \( \mathcal{B}(K/F) \), since \( D \) is not split by \( C \) or \( K_\ast \). We have \( [A : F] = p^2 = [B : F] \), and since neither \( A \) nor \( B \) are matrices over \( F \), they must both be division alge-

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bras [1, Theorem 3.18, p. 43]. By hypothesis $A \otimes B \cong D \otimes \mathfrak{m}$ where $\mathfrak{m}$ is a full matrix ring over $F$; however, a dimension count shows $\mathfrak{m} = F$, and $A \otimes B \cong D$. Thus we can consider $A$ and $B$ as division subalgebras of $D$, both with center $F$ and each the centralizer of the other [1, Theorem 4.13, p. 53].

Now $A$ is a cross-product, indeed a cyclic algebra: $A = K_s \oplus yK_s \oplus \cdots \oplus y^{p-1}K_s$, where $y$ satisfies the following properties: $y^{-1}ky = k^\sigma (\sigma \in \mathcal{O}(K_s/F), \sigma \neq 1)$ and $yp \in F$ [1, p. 74]. Considered as an element of $D$, $y$ must commute with $B$ and hence with $C \subset B$. Thus $C(y)$ is a field contained in $D$. On the one hand, $C(y)$ must be a bigger field than $C$, since if $y$ were in $C$ it would commute with $K_s$; on the other hand, since $yp \in F$ and $C_p = F$, $y \in C(y)$ must be in $C$. This contradiction proves the theorem.

3. Construction of $D$. We must exhibit: a field $F$, a division algebra $D$ with center $F$ and degree $p^2$ over $F$ (i.e. $[D:F] = p^4$); a subfield of $D$, $K_s$, containing $F$ and cyclic over $P$; a field $C$ such that $F \subset C \subset D$ and $C_p = F$, such that $C$ and $K_s$ commute elementwise.

Let $F = GF(p^n)(\pi)$ where $\pi$ is an indeterminate over $GF(p)$. Let $R$ be the (unique) cyclic extension of $GF(p)$ of degree $p^2$. Set $L = R(\pi)$; $L$ is a cyclic extension of $F$ of degree $p^2$. Let $\sigma$ be the generating automorphism of $g(L/F); \sigma^p = 1$. Finally, form $D = (L, \sigma, \pi)$ [1, p. 74].

One sees immediately via polynomial degree considerations that the indeterminate $\pi$ is not a norm from $K_s = \{k \in L; k^\sigma = k\}$. This suffices to show that $D$ is a division algebra [1, Theorem 7.19, p. 98]. Note that the field $K_s$ is cyclic of degree $p$ over $F$.

We construct $C$ as follows: by definition of $D$ there exists an element $y \in D$ such that $y^p = \pi$ ($y$ induces the automorphism $\sigma$ in $L \subset D = (L, \sigma, \pi)$). Let $C = F(y^p)$; then, since $(y^p)^p = \pi$, $C_p = F$. Note that $C$ and $K_s$ commute, since $y^p$ and $K_s$ do; thus $K = C \otimes K_s$ is a subfield of $D$.

Bibliography