ON TOPOLOGY INDUCED BY MEASURE

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Let \((X, \mathfrak{M}, \mu)\) be a complete \(\sigma\)-finite measure space with \(\mu(X) > 0\). The purpose of this note is to record the existence of a topology on \(X\) with the property that each measurable (extended) real-valued function \(f\) on \(X\) is equal almost everywhere to a unique continuous (extended) real-valued function \(f^*\) on \(X\). The mapping \(f \mapsto f^*\) provides a "lifting," in the sense that it preserves the algebraic operations (where they are defined). The Stone-Čech compactification of this topology is also discussed.

If \(\mathfrak{M}\) denotes the \(\sigma\)-ideal in \(\mathfrak{M}\) of the sets of measure zero, then by a result of Maharam ([3]; see also [6]), there exists a mapping \(\phi: \mathfrak{M}/\mathfrak{N} \to \mathfrak{N}\) such that

1. \(\phi(0) = \emptyset\) and \(\phi(1) = X\),
2. \(\phi(p \cap q) = \phi(p) \cap \phi(q)\),
3. \(\phi(p \cup q) = \phi(p) \cup \phi(q)\),
4. \(\phi(p) \in \mathfrak{N}\),

for all \(p, q \in \mathfrak{M}/\mathfrak{N}\). By (1) and (2) the sets \(\phi(p)\) provide a basis for a topology on \(X\), and this is the topology in question.

**Theorem.** Each continuous function on \(X\) is measurable. For each measurable function \(f\) on \(X\), there exists a unique continuous function \(f^*\) on \(X\) which agrees almost everywhere with \(f\). The mapping \(f \mapsto f^*\) preserves the algebraic operations (where they are defined).

**Proof.** For the first statement, it is sufficient to show that each open set is measurable. The following proof of this is taken from [3]. Any basic open set is measurable since the range of \(\phi\) is in \(\mathfrak{M}\). Consider any family \(\{\phi(p_a)\}\) of basic open sets. Since the measure space is \(\sigma\)-finite, \(\mathfrak{M}/\mathfrak{N}\) satisfies the countable chain condition and is a complete lattice. Hence \(p = \bigcup p_a\) exists, and there is a sequence \(\alpha_1, \alpha_2, \ldots\) of indices with \(p = \bigcup p_{\alpha_i}\). Since \(\phi\) is monotone (by (2) or (3)),

\[
\bigcup \phi(p_{\alpha_i}) \subset \bigcup \phi(p_a) \subset \phi(p).
\]

But \(\phi(p_{\alpha_i}) \in p_{\alpha_i}\), by (4), so \(\bigcup \phi(p_{\alpha_i}) \in P_{\alpha_i} = p\). Also \(\phi(p) \in \mathfrak{N}\), hence \(\phi(p) - \bigcup \phi(p_{\alpha_i})\) is a null set, and therefore \(\bigcup \phi(p_a)\) is measurable.

Now let \(f\) be a measurable function on \(X\), and define

\[
f^*(x) = \sup \{r \mid x \in \phi[f^{-1}(-\infty, r)]\}
\]

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for each $x \in X$. Here we are denoting by $[A]$ the element of $\mathcal{M}/\mathcal{N}$ corresponding to $A \in \mathcal{M}$. It is easy to verify that

$$f^*-1[-\infty, r) = \bigcup_{s<r} \phi[f^{-1}[-\infty, s)]$$

and

$$f^*-1(r, \infty] = \bigcup_{s<r} (X - \phi[f^{-1}(-\infty, s)].$$

Assumptions (1)–(3) imply that if $p'$ is the complement of $p$, then $\phi(p')$ is the complement of $\phi(p)$, and therefore $f^*$ is continuous. Since the above unions need only be extended over rational $s$, the symmetric difference of $f^*-1[-\infty, r)$ and $f^{-1}[-\infty, r)$ is a null set for each real $r$. Consequently $f$ and $f^*$ differ only on a null set. For the uniqueness, it is sufficient to observe that each nonvoid open set contains a basic open set $\phi(p)$ with $p \neq 0$, and therefore has positive measure. That the algebraic operations are preserved is an immediate consequence of the uniqueness.

Our second observation concerns the Stone-Cech compactification of $X$. In order to discuss this, we introduce the Stone space $Y$ of the complete Boolean algebra $\mathcal{M}/\mathcal{N}$. The points of $Y$ are the ultrafilters (maximal dual ideals) $\mathcal{F}$ of $\mathcal{M}/\mathcal{N}$, and the basic open sets in $Y$ are the sets

$$\gamma(p) = \{\mathcal{F} \mid p \in \mathcal{F}\}$$

as $p$ varies over $\mathcal{M}/\mathcal{N}$. With this topology $Y$ is an extremally disconnected compact Hausdorff space, and the lattice $\mathcal{M}/\mathcal{N}$ is isomorphic to the lattice of all open and closed subsets of $Y$. For each $x \in X$ define

$$F(x) = \{p \mid p \in \mathcal{M}/\mathcal{N} \text{ and } x \in \phi(p)\}.$$ 

A similar construction is used by Donoghue in [1].

**Lemma.** $F(x)$ is an ultrafilter in $\mathcal{M}/\mathcal{N}$. The mapping $F: X \to Y$ satisfies $\phi = F^{-1} \circ \gamma$, and consequently the topology in $X$ is the weakest such that $F$ is continuous.

**Proof.** Since $\phi(0) = \emptyset$ by (1), we have $0 \notin F(x)$. On the other hand $\phi(1) = X$, so $1 \in F(x)$ for all $x$. If $p, q \in F(x)$, then $x \in \phi(p) \cap \phi(q) = \phi(p \cap q)$ by (2), so $p \cap q \in F(x)$. Finally, if $p \in \mathcal{M}/\mathcal{N}$ and $p'$ is its complement, then

$$\phi(p) \cup \phi(p') = \phi(p \cup p') = \phi(1) = X$$

by (3) and (1). Thus either $x \in \phi(p)$ or $x \in \phi(p')$, so that either
\( p \in F(x) \) or \( p' \in F(x) \). Therefore \( F(x) \) is an ultrafilter, and \( F(x) \subseteq Y \).

Concerning the second assertion of the lemma, for each \( p \in \mathfrak{M} / \mathfrak{M} \) and \( x \in X \) the following statements are equivalent: \( x \in (F^{-1} \circ \gamma)(p) \), \( F(x) \subseteq \gamma(p) \), \( p \in F(x) \), and \( x \in \phi(p) \).

We remark that \( X \) is a completely regular space, and that it is Hausdorff when \( F \) is one-to-one.

**Theorem.** Let \( K \) be a compact Hausdorff space, and let \( \alpha: X \to K \) be continuous. Then there exists a unique continuous \( \bar{\alpha}: Y \to K \) such that \( \alpha = \alpha \circ F \).

**Proof.** We begin by observing that \( F(X) \) is dense in \( Y \). For if the basic open set \( \gamma(p) \) is nonempty, then \( p \neq 0 \), \( \phi(p) \) is nonempty, and \( F(x) \subseteq \gamma(p) \) for any \( x \in \phi(p) \). Moreover, \( Y \) is extremally disconnected, so that \( F(X) \) is \( C^* \)-embedded [2, p. 96]. Therefore \( Y \) is the Stone-
Čech compactification of \( F(X) \).

Next we note the existence of a continuous function \( \beta: F(X) \to K \) such that \( \alpha = \beta \circ F \). It follows easily from the lemma that \( F(x_1) = F(x_2) \) implies \( \alpha(x_1) = \alpha(x_2) \), so that the function \( \beta: F(x) \to \alpha(x) \) is well defined. The lemma also implies that \( F \) is open in its range, so that if \( U \) is an open subset of \( K \), then \( \beta^{-1}(U) = F(\alpha^{-1}(U)) \) is open. Thus \( \beta \) is continuous.

Finally, by the first paragraph of the proof, \( \beta \) has a continuous extension \( \bar{\alpha} \) to all of \( Y \), and clearly \( \alpha = \bar{\alpha} \circ F \). The density of \( F(X) \) implies the uniqueness assertion, and completes the proof.

**Remarks.**

1. A proof of the above theorem may be based on the following construction. If \( g \) is a continuous (extended) real-valued function on \( X \), let

\[
\tilde{g}(\mathfrak{f}) = \sup \{ r \mid \mathfrak{f} \subseteq \gamma[g^{-1}[-\infty, r]] \}
\]

for each \( \mathfrak{f} \subseteq Y \). Then \( \tilde{g} \) is a continuous function on \( Y \) satisfying \( g = \tilde{g} \circ F \). If \( g \) is merely measurable, then \( g^* = \tilde{g} \circ F \) is the continuous function constructed in the first theorem.

2. The mapping \( f \mapsto (f^*)^- \) from \( L^\infty(X, \mathfrak{M}, \mu) \) into \( C(Y) \) is easily seen to be an isometric isomorphism, and is in fact the Gelfand transform.

3. In [7] the topology with basis consisting of the sets \( \phi(p) - N \), where \( N \subseteq \eta \), is considered. This is stronger than the above topology, but admits no more continuous functions.

4. Those measure spaces for which a mapping \( \phi \) satisfying (1)-(4) exists are characterized in [4]. If the space is not \( \sigma \)-finite, then a continuous function may fail to be measurable, but the remainder of the
first theorem is valid in this context. For the second theorem the completeness of $\mathcal{M}/\mathcal{N}$ appears essential.

REFERENCES