ON CUBIC TRANSFORMATIONS OF
ORTHOGONAL POLYNOMIALS

PIERRE BARRUCAND AND DAVID DICKINSON

After investigating generalizations of the quadratic transformation
relating Hermite and Laguerre polynomials

\[ H_{2n}(x) = (-1)^n 2^n n! L_{n-1/2}(-x^2), \]

T. S. Chihara asked [1] for an example of a pair of orthogonal
polynomial sets \{R_n(x)\} and \{S_n(x)\} such that \(R_{3n}(x) = S_n(z)\) where
\(z\) is a cubic in \(x\). It is the purpose of this paper to exhibit such ex-
amples.

Let \{S_n(z)\} be a set of monic polynomials with \(S_n(-z) = (-1)^n S_n(z)\). For this set to be a set of orthogonal polynomials, it is
both necessary and sufficient (Shohat [3, pp. 454 and 456]) that there
exist a sequence of positive numbers \{B_n\} such that

\[ zS_n(z) = S_{n+1}(z) + B_n S_{n-1}(z), \quad n = 1, 2, \ldots , \]
or equivalently,

\[ (x^3 + \beta x)S_n(x^3 + \beta x) = S_{n+1}(x^3 + \beta x) + B_n S_{n-1}(x^3 + \beta x), \]

\[ n = 1, 2, \ldots . \]

We look for a monic set of orthogonal polynomials \{R_n(x)\} with
\(R_n(-x) = (-1)^n R_n(x)\) and with a subset \{R_{3n}(x)\} that satisfies a
recurrence relation with the same coefficients,

\[ (x^3 + \beta x)R_{3n}(x) = R_{3n+3}(x) + B_n R_{3n-3}(x), \quad n = 1, 2, \ldots . \]

Such a set of orthogonal polynomials exists if (3) can be formed by
iterating

\[ xR_n(x) = R_{n+1}(x) + b_n R_{n-1}(x), \quad n = 1, 2, \ldots , \]

the condition (with \(b_n > 0, n = 1, 2, \ldots\)) that is equivalent to the
condition that \{R_n(x)\} is an orthogonal set.

Iterating (4), we may write successively

\[ xR_{3n}(x) = R_{3n+1}(x) + b_{3n} R_{3n-1}(x), \]

\[ x^2 R_{3n}(x) = (R_{3n+2}(x) + b_{3n+1} R_{3n}(x)) + b_{3n} (R_{3n}(x) + b_{3n-1} R_{3n-2}(x)), \]

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\[
x^3 R_{3n}(x) = (R_{3n+3}(x) + b_{2n+2} R_{3n+1}(x)) \\
+ (b_{3n+1} + b_{3n}) (R_{3n+1}(x) + b_{3n} R_{3n-1}(x)) \\
+ b_{3n} b_{3n-1} (R_{3n-1}(x) + b_{3n-2} R_{3n-3}(x))
\]

and thus

\[
(x^3 + \beta x) R_{3n}(x) = R_{3n+3}(x) \\
+ (b_{3n+2} + b_{3n+1} + b_{3n} + \beta) R_{3n+1}(x) \\
+ (b_{3n+1} + b_{3n} + b_{3n-1} + \beta) b_{3n} R_{3n-1}(x) \\
+ b_{3n} b_{3n-1} b_{3n-2} R_{3n-3}(x), \quad n = 1, 2, \ldots
\]

is the same recurrence relation as (3) provided \(\beta\) and the \(b_n\) are such that

\[
\begin{align*}
&b_{3n+2} + b_{3n+1} + b_{3n} + \beta = 0, \quad n = 1, 2, \ldots, \\
&b_{3n+1} + b_{3n} + b_{3n-1} + \beta = 0, \quad n = 1, 2, \ldots, \\
&b_{3n} b_{3n-1} b_{3n-2} = B_n, \quad n = 1, 2, \ldots.
\end{align*}
\]

Thus, assuming these three conditions (5) on the recurrence relations (1) and (4), the polynomial sets \(\{S_n(x^3 + \beta x)\}\) and \(\{R_{3n}(x)\}\) satisfy the same recurrence relations ((2) or (3)) and if we further place the initial restrictions

\[
\begin{align*}
&S_0(x^3 + \beta x) = R_0(x) \\
&S_1(x^3 + \beta x) = R_3(x),
\end{align*}
\]

then we will have

\[
S_n(x^3 + \beta x) = R_{3n}(x), \quad n = 0, 1, 2, \ldots.
\]

Condition (6) is satisfied because our assumption that polynomial sets are monic implies \(S_0(z) = R_0(x) = 1\). Since \(S_1(z) = z\), condition (7) becomes \(x^3 + \beta x = R_3(x)\) or, because \(R_3(x) = x R_2(x) - b_2 R_1(x) = x^2 - (b_1 + b_2)x\), we may replace (7) by the condition

\[
b_1 + b_2 + \beta = 0.
\]

Thus, after combining (8) and the three conditions (5), we have established the

**Theorem.** If \(\{S_n(z)\}\) is a set of orthogonal polynomials with \(S_0(z) = 1, S_1(z) = z\) and

\[
z S_n(z) = S_{n+1}(z) + B_n S_{n-1}(z), \quad n = 1, 2, \ldots
\]
and if there exist numbers $\beta$ and $b$ and a sequence of positive numbers \( \{b_n\} \) such that the four conditions

\[
\begin{align*}
&b_1 + b + \beta = 0, \\
&b_{3n-1} = b, \quad n = 1, 2, \ldots, \\
&b_{3n+1} + b_3 + b + \beta = 0, \quad n = 1, 2, \ldots, \\
&b_{3n}b_{3n-2} = B_n, \quad n = 1, 2, \ldots,
\end{align*}
\]

are met, then the polynomials \( \{R_n(x)\} \) defined by \( R_0(x) = 1, R_1(x) = x, \) and

\[
xR_n(x) = R_{n+1}(x) + b_nR_{n-1}(x), \quad n = 1, 2, \ldots
\]

form an orthogonal set with the property

\[
S_n(x^3 + \beta x) = R_{3n}(x), \quad n = 0, 1, \ldots.
\]

After this theorem, one naturally inquires about expressing the polynomials of the sets \( \{R_{3n+1}(x)\} \) and \( \{R_{3n+2}(x)\} \) in terms of the polynomials of the set \( \{S_n(x^3+\beta x)\} \). We can express \( R_{3n+1}(x) \) in terms of \( R_{3n}(x) \) and \( R_{3n+3}(x) \) by successively combining recurrence relations:

\[
\begin{align*}
xR_{3n+1}(x) &= R_{3n+2}(x) + b_{3n+1}R_{3n}(x), \\
x^2R_{3n+1}(x) &= (R_{3n+3}(x) + b_{3n+2}R_{3n+1}(x)) + xb_{3n+1}R_{3n}(x), \\
(x^2 - b_{3n+2})R_{3n+1}(x) &= R_{3n+3}(x) + xb_{3n+1}R_{3n}(x).
\end{align*}
\]

Thus we are led to the

**Corollary.** For polynomial sets meeting the hypotheses of the theorem,

\[
\begin{align*}
R_{3n+1}(x) &= (x^2 - b)^{-1}(S_{n+1}(x^3 + \beta x) + xb_{3n+1}S_n(x^3 + \beta x)), \\
&\quad n = 0, 1, \ldots, \\
R_{3n+2}(x) &= (x^2 - b)^{-1}(xS_{n+1}(x^3 + \beta x) + b_{3n+2}b_{3n+1}S_n(x^3 + \beta x)), \\
&\quad n = 0, 1, \ldots.
\end{align*}
\]

A particularly simple set of constants meeting the conditions of the theorem is \( b_n = B_n = 1 \) for \( n > 1, \beta = -3, \) and \( b_1 = B_1 = 2. \) For this choice of constants, the two sets \( \{R_n(x)\} \) and \( \{S_n(x)\} \) are identical and \( R_{3n}(x) = R_n(x^3 - 3x), n = 0, 1, \ldots. \) The recurrence relation

\[
\begin{align*}
xR_n(x) &= R_{n+1}(x) + R_{n-1}(x), \quad n = 1, 2, \ldots, \\
xR_1(x) &= R_1(x) + 2R_0(x)
\end{align*}
\]
and the polynomials \( \{ R_n(x) \} \) when transformed by \( R_0(2x) = T_0(x) \) and \( R_n(2x) = 2T_n(x) \), \( n = 1, 2, \cdots \), lead us to the nonmonic set \( \{ T_n(x) \} \) defined by

\[
T_0(x) = 1, \quad T_1(x) = x,
\]

\[
2xT_n(x) = T_{n+1}(x) + T_{n-1}(x), \quad n = 1, 2, \cdots.
\]

But this is a definition of the Tchebichef polynomials of the first kind \([2, (10.11)]\). Thus we have shown that for the Tchebichef polynomials of the first kind \( \{ T_n(x) \} \),

\[
T_{3n}(x) = T_n(4x^3 - 3x), \quad n = 0, 1, \cdots,
\]

\[
T_{3n+1}(x) = (4x^2 - 1)^{-1}(T_{n+1}(4x^3 - 3x) + 2xT_n(4x^3 + 3x)),
\]

(10)

\[
T_{3n+2}(x) = (4x^2 - 1)^{-1}(2xT_{n+1}(4x^3 - 3x) + T_n(4x^3 - 3x)),
\]

\( n = 0, 1, \cdots. \)

The cubic transformation (10), which may be written as \( T_{3n}(x) = T_n(T_3(x)) \), is a special case of the transformation of arbitrary degree

\[
T_{mn}(x) = T_m(T_n(x)), \quad m, n = 0, 1, \cdots,
\]

that is an easy consequence of the definition \( T_n(\cos \theta) = \cos n\theta \).

For a less trivial example of the theorem, let the \( S_n(z) \) be the monic Legendre polynomials defined in terms of the ordinary Legendre polynomials \( P_n(z) \) by \( 2^n(1/2)_n(n!)^{-1}S_n(z) = P_n(z), \ n = 0, 1, \cdots \), or equivalently, defined directly by

\[
S_0(z) = 1, \quad S_1(z) = z,
\]

\[
xS_n(z) = S_{n+1}(z) + \frac{n^2}{(2n + 1)(2n - 1)} S_{n-1}(z), \quad n = 1, 2, \cdots.
\]

Here we have

\[
B_n = \frac{n^2}{(2n + 1)(2n - 1)}, \quad n = 1, 2, \cdots,
\]

and the remaining numbers of the theorem may be chosen as

\[
b_{3n} = \frac{n}{2n + 1}, \quad n = 1, 2, \cdots,
\]

\[
b_{3n+1} = \frac{n + 1}{2n + 1}, \quad n = 0, 1, \cdots.
\]
That these polynomials \( \{ R_n(x) \} \) where
\[
2^n(1/2)_n(n!)^{-1} R_3n(x) = P_n(x^3 - 2x),
\]
\[
2^{n+1}(1/2)_{n+1}((n + 1)!)^{-1} R_{3n+1}(x) = (x^2 - 1)^{-1}(P_{n+1}(x^3 - 2x) + xP_n(x^3 - 2x)),
\]
\[
2^{n+1}(1/2)_{n+1}((n + 1)!)^{-1} R_{3n+2}(x) = (x^2 - 1)^{-1}(xP_{n+1}(x^3 - 2x) + P_n(x^3 - 2x))
\]
are orthogonal over a bounded domain follows from the boundedness of the coefficients in the recurrence relation (4) by a theorem due to Shohat ([3, p. 458]). Since the zeroes of the polynomials of the set \( \{ P_n(z) \} \) are dense in the domain \( |z| \leq 1 \), the zeroes of the polynomials of the set \( \{ P_n(x^3 - 2x) \} \) (which are the same as the zeroes of the set \( \{ R_{3n}(x) \} \), \( n = 0, 1, \ldots \)) are dense in the domain \( D \) consisting of the points \( x \) such that \( |x^3 - 2x| \leq 1 \). This \( D \) is the union of the three disjoint intervals \( ((-1 - (5)^{1/2})/2, -1), ((1 - (5)^{1/2})/2, (-1+(5)^{1/2})/2), \) and \( (1, (1+(5)^{1/2})/2) \). It follows from Szegö ([4, Theorem 6.1.1]) that this domain \( D \) is precisely the "true" domain of orthogonality of this set \( \{ R_n(x) \} \).