SPACES OF CONSTANCY OF CURVATURE OPERATORS[1]

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1. **Introduction.** There are three kinds of Riemannian manifolds whose study is facilitated by the fact that their curvature operators have a particularly nice form; these are flat spaces, spaces of constant [curvature, and spaces of constant] holomorphic curvature. It is natural, therefore, on a general Riemannian manifold M to study the distribution which assigns to each $m \in M$ the subspace of the tangent space M_m to M at m on which the curvature operator behaves like one of the above types. For example, let $\mathcal{R}(m) = \{x \in M_m \mid R_{xy} = 0 \text{ for all } y \in M_m\}$, where R_{xy} denotes the curvature operator. Chern and Kuiper [1] have proved that the distribution $m \to \mathcal{R}(m)$ is integrable and its integral manifolds are flat. Maltz [4] has shown that the integral manifolds are totally geodesic and investigated their completeness properties.

The subspace $\mathcal{R}(m)$ of M_m is called the *space of nullity* of the curvature operator at m. Similarly we define the *space of constancy* $\mathcal{R}_K(m)$ and the *space of holomorphic constancy* $\mathcal{H}_K(m)$ of the curvature operator at m. Here $\mathcal{R}_K(m)$ is the subspace of M_m on which the curvature operator has constant curvature K, and $\mathcal{H}_K(m)$ is the subspace on which it has constant holomorphic curvature $4K$. We prove that the distributions $m \to \mathcal{R}_K(m)$ and $m \to \mathcal{H}_K(m)$ are integrable. The integral submanifolds are totally geodesic, have constant or constant holomorphic curvature and possess the same completeness properties as in the flat case. Ôtsuki [6] has also considered the spaces of constancy $\mathcal{R}_{K(m)}$.

2. **Riemannian tensors.** Let M be a Riemannian manifold of class C^∞; we denote by $\mathfrak{F}(M)$ the ring of differentiable real-valued functions on M and by $\mathfrak{X}(M)$ its derivation Lie algebra, which consists of the vector fields on M. The metric tensor field will be denoted by $\langle\, ,\rangle$, the Riemannian connection by ∇_X $(X \in \mathfrak{X}(M))$, and the curvature operator by $R_{XY}(X, Y \in \mathfrak{X}(M))$. If M is almost complex $J\colon \mathfrak{X}(M) \to \mathfrak{X}(M)$ will denote the almost complex structure. Since $\langle\, ,\rangle$, R, and J are tensor fields, they determine tensors on each tangent space, which we denote by the same symbols. In order to unify our proofs we consider a special class of (1, 3) tensor fields.

Received by the editors October 4, 1965.

[1] This work was supported in part by the National Science Foundation under Grant GP-3990.

(2.1) DEFINITION. A *Riemannian tensor field* on M is a tensor field A of type (1, 3), which for X, $Y\in\mathfrak{X}(M)$ we regard as an $\mathfrak{F}(M)$-linear map $A_{XY}\colon \mathfrak{X}(M)\to\mathfrak{X}(M)$. It is required to have the following properties:

(2.1.1) $A_{XY}=-A_{YX}$,
(2.1.2) $\langle A_{XY}(Z), W\rangle=-\langle A_{XY}(W), Z\rangle$,
(2.1.3) $\mathfrak{S}A_{XY}(Z)=0$,
(2.1.4) $\mathfrak{S}\nabla_X(A)_{YZ}=0$,

for X, Y, Z, $W\in\mathfrak{X}(M)$, where $\mathfrak{S}$ denotes the cyclic sum. If $\nabla_X(A)_{YZ}=0$ for all X, Y, $Z\in\mathfrak{X}(M)$ we say that A is *parallel*, and if $\nabla_X(A)=\alpha(X)A$ for some 1-form α we say that A is *recurrent*.

It is easy to see that (2.1.1)–(2.1.3) imply

(2.1.5) $\langle A_{XY}(Z), W\rangle=\langle A_{ZW}(X), Y\rangle$ for X, Y, Z, $W\in\mathfrak{X}(M)$.

If $\langle A_{XY}(X), Y\rangle=0$ for all X, $Y\in\mathfrak{X}(M)$, then $A=0$ (see Helgason [**3**, p. 68]). Also, it is clear that the Riemannian tensor fields are closed under addition and multiplication by real numbers.

(2.2) LEMMA. *Let A be a Riemannian tensor. Then for X, Y, $Z\in\mathfrak{X}(M)$ we have*

$$\mathfrak{S}\{[\nabla_X, A_{YZ}] - A_{[X,Y]Z}\} = 0.$$

PROOF. We have $\nabla_X(Y)-\nabla_Y(X)=[X, Y]$ for X, $Y\in\mathfrak{X}(M)$, and so

$$\begin{aligned} 0 = \mathfrak{S}\nabla_X(A)_{YZ} &= \mathfrak{S}\{[\nabla_X, A_{YZ}] - A_{\nabla_X(Y)Z} - A_{Y\nabla_Y(Z)}\} \\ &= \mathfrak{S}\{[\nabla_X, A_{YZ}] - A_{[X,Y]Z}\}. \end{aligned}$$

(2.3) DEFINITION. Let $m\in M$. We define

$$\mathfrak{A}(m) = \{x \in M_m \mid A_{xy} = 0 \text{ for all } y \in M_m\},$$

and we denote by $\mathfrak{A}$ the distribution $m\to\mathfrak{A}(m)$. (Here A_{xy} is the operator on M_m determined by A.) We call $\mathfrak{A}(m)$ the *space of nullity* of A at m, $\mathfrak{A}$ the *field of nullity* of A, and dim $\mathfrak{A}(m)$ the *index of nullity* of A at m.

(2.4) THEOREM. *Let U be an open subset of M on which the index of nullity of A is constant. Then the distribution $\mathfrak{A}$ is integrable on U.*

PROOF. Let X and Y be a vector fields in $\mathfrak{A}$. From (2.2) it follows that $[X, Y]$ is in $\mathfrak{A}$.

It is clear that the curvature operator is a Riemannian tensor field; however, there are two other Riemannian tensors that we shall be particularly interested in, namely B and D, defined as follows:

$$B_{XY}(Z) = R_{XY}(Z) - K(\langle X, Z\rangle Y - \langle Y, Z\rangle X).$$

$$D_{XY}(Z) = R_{XY}(Z) - K(\langle X, Z\rangle Y - \langle Y, Z\rangle X + \langle JX, Z\rangle JY - \langle JY, Z\rangle JX + 2\langle JX, Y\rangle JZ),$$

for X, Y, $Z \in \mathfrak{X}(M)$, where K is a constant. The latter tensor field is defined if M is almost complex. It is not hard to verify that B is Riemannian, and if M is Kählerian, D is Riemannian. It can be shown [**3**], [**7**] that $B=0$ if and only if M has constant curvature K, and that $D=0$ if and only if M has constant holomorphic curvature $4K$.

We denote by $\mathcal{R}_K$ and $\mathcal{H}_K$ the fields of nullity of B and D respectively, and we call $\mathcal{R}_K(m)$ and $\mathcal{H}_K(m)$ the *spaces of constancy and holomorphic constancy* of the curvature operator at m. We shall be concerned only with the tensor fields B and D; however, there are other interesting Riemannian tensor fields. For example, if M is locally symmetric (i.e., the curvature operator is parallel), the Weyl conformal tensor field is a parallel Riemannian tensor field, and by (2.4) its field of nullity is integrable.

3. **Local properties of the integral manifolds.** Let L be a Riemannian manifold isometrically imbedded in another Riemannian manifold M. Let $\overline{\mathfrak{X}}(L) = \{X | L | X \in \mathfrak{X}(M)\}$; then we write $\overline{\mathfrak{X}}(L) = \mathfrak{X}(L) \oplus \mathfrak{X}(L)^\perp$ where $\mathfrak{X}(L)^\perp$ is the collection of vector fields normal to L. Let $P: \overline{\mathfrak{X}}(L) \to \mathfrak{X}(L)$ be the natural projection. For X, $Y \in \mathfrak{X}(L)$ we denote the Riemannian connection and curvature operator of L by δ_X and r_{XY} respectively. The *configuration tensor* [**2**] of L in M is an $\mathfrak{F}(M)$-linear map $t: \mathfrak{X}(L) \times \overline{\mathfrak{X}}(L) \to \overline{\mathfrak{X}}(L)$ defined by $t_X(Y) = \nabla_X(Y) - \delta_X(Y)$ $(X, Y \in \mathfrak{X}(L))$ and $t_X(Z) = P\nabla_X(Z)$ $(X \in \mathfrak{X}(L), Z \in \mathfrak{X}(L)^\perp)$. Then $t_X(\mathfrak{X}(L)) \subseteq \mathfrak{X}(L)^\perp$, $t_X(\mathfrak{X}(L)^\perp) \subseteq \mathfrak{X}(L)$ for $X \in \mathfrak{X}(L)$, $t_X(Y) = t_Y(X)$ for X, $Y \in \mathfrak{X}(L)$ and $\langle t_X(Z), W\rangle = -\langle t_X(W), Z\rangle$ for W, $X \in \mathfrak{X}(L)$, $Z \in \mathfrak{X}(L)^\perp$. The configuration tensor vanishes if it vanishes on either $\mathfrak{X}(L)$ or $\mathfrak{X}(L)^\perp$, and so it is equivalent to the second fundamental form, which in our terminology would be the map $X \to t_X(Z)$ for $X \in \mathfrak{X}(L)$, $Z \in \mathfrak{X}(L)^\perp$.

We now prove that the field of nullity of a Riemannian tensor is totally geodesic. First we state a lemma, the proof of which is obvious.

(3.1) LEMMA. *Let L be an integral manifold of $\mathfrak{A}$. Then if X, Y, $Z \in \mathfrak{X}(L)^\perp$, $A_{XY}(Z) \in \mathfrak{X}(L)^\perp$.*

(3.2) THEOREM. *Let L be an integral manifold of $\mathfrak{A}$; then L is totally geodesic.*

PROOF. Let $X \in \mathfrak{X}(L)$ and Y, Z, $U \in \mathfrak{X}(L)^\perp$. We first show that $t_X(A_{YZ}(U)) = 0$. Since $A_{YZ}(U) \in \mathfrak{X}(L)^\perp$ we have

$$\begin{aligned} P\mathfrak{S}_{XYZ}\nabla_X(A_{YZ}(U)) &= P\{\nabla_X(A_{YZ}(U)) + \nabla_Z(A_{XY}(U)) + \nabla_Y(A_{ZX}(U))\} \\ &= t_X(A_{YZ}(U)). \end{aligned}$$

On the other hand by (2.2) and (3.1)

$$\begin{aligned} P\mathfrak{S}_{XYZ}\nabla_X(A_{YZ}(U)) &= P\mathfrak{S}\{A_{YZ}(\nabla_X(U)) + A_{[X,Y]Z}(U)\} \\ &= A_{YZ}P\nabla_X(U) + A_{P[X,Y]Z}(U) + A_{P[Z,X]Y}(U) \\ &= 0. \end{aligned}$$

Next we let $W\in\mathfrak{X}(L)$. Then $t_X(W)\in\mathfrak{X}(L)^\perp$ and so by (3.1) $A_{YZ}(t_X(W))\in\mathfrak{X}(L)^\perp$; however, $\langle A_{YZ}(t_X(W)),\ U\rangle=\langle W,\ t_X(A_{YZ}(U))\rangle =0$ and so $t_X(W)\in\mathfrak{X}(L)$. Hence $t_X(W)=0$; this proves that L is totally geodesic.

(3.3) COROLLARY. *Suppose L is an integral manifold of $\mathcal{R}_K$ or $\mathcal{H}_K$. Then L has constant curvature K or constant holomorphic curvature $4K$. If M is a Kähler manifold and M is an integral manifold of $\mathcal{H}_K$, then L is a Kähler submanifold of M.*

PROOF. The first statement follows from the Gauss equation (see [2]):

$$PR_{XY} = r_{XY} - [t_X, t_Y].$$

For the last statement let $X\in\mathfrak{X}(L)$, $Y\in\overline{\mathfrak{X}}(L)$. Then

$$D_{JXY} = JD_{XY} = 0,$$

and so $JX\in\mathfrak{X}(L)$; hence L is a Kähler submanifold of M.

4. **The completeness of the integral manifolds.** Let G be the set on which the index of nullity μ assumes its minimum value λ.

(4.1) PROPOSITION. *The function μ is upper semicontinuous, and the set G is open.*

PROOF. It suffices to prove that for any $m\in M$ there exists a neighborhood U of m such that $\mu(p)\leqq\mu(m)$ for $p\in U$, but this is obvious.

We shall need the following lemma. Let $n=$dimension of M.

(4.2) LEMMA. *Let γ: $[0, b)\to L$ be a unit speed geodesic in an integral manifold L of $\mathfrak{A}$ in G. Then there exists a frame field $\{e^1, \cdots, e^n\}$ on γ such that:*

(4.2.1) $e^i_{\gamma(t)}\in\mathfrak{A}(\gamma(t))$ $(1\leqq i\leqq\lambda,\ t\in[0, b))$.

(4.2.2) $\gamma'(t)=e^1_{\gamma(t)}$ $(t\in[0, b))$.

(4.2.3) *The frame field is parallel on γ.*

PROOF. We may assume (4.2.1)–(4.2.3) hold for $t=0$. The frame field is defined at an arbitrary $t\in[0, b)$ by parallel translation. It is obvious that (4.2.2) and (4.2.3) are satisfied, and (4.2.1) holds because the parallel translation takes place along the submanifold L.

(4.3) THEOREM. *Assume M is complete and that A is recurrent. Then each integral manifold L of $\mathfrak{A}$ in G is complete.*

PROOF. If $\lambda=n$ the proof is trivial, so we assume $\lambda<n$. Let γ: $[0, b) \to L$ be a unit speed geodesic. Since M is complete we may extend γ to a geodesic γ: $[0, \infty)\to M$. Let $\{e^1, \cdots, e^n\}$ be a frame field on $\gamma|[0, b)$ which satisfies (4.2.1)–(4.2.3). Then we may define $\{e_1, \cdots, e_n\}$ at $\gamma(b)$ by parallel translation; (4.2.1)–(4.2.3) now hold for $\gamma|[0, b]$. We extend each e^i to a vector field E^i defined on a neighborhood N of $\gamma|[0, b]$. Also, let X, Y be vector fields on N such that $X_{\gamma(t)}$, $Y_{\gamma(t)}\in\mathfrak{A}(\gamma(t))^\perp$ for $t\in[0, b)$ and X, Y are parallel on $\gamma|[0, b]$.

Let $\lambda+1\leqq p, q, r\leqq n$; we define Φ_{pq}: $[0, b]\to R$ and Γ_{pq}: $[0, b]\to R$ by

$$\Phi_{pq} = \langle A_{E^pE^q}(X), Y\rangle \circ \gamma$$

By (2.1.4) and (4.2.1)–(4.2.3) it follows that

$$\Phi'_{pq} = 0. \tag{4.3.1}$$

Since the matrix (Φ_{pq}) is nonzero at 0, it follows from the theory of ordinary differential equations that (Φ_{pq}) cannot vanish at b. The vector fields X and Y are arbitrary and so $\mu(\gamma(b))=\lambda$. Therefore $\gamma(b)\in G$ and so there exists $c>b$ such that $\gamma([0, c))\subset G$. Hence every geodesic in L is infinitely extendable (in L) and so L is complete.

5. **Some examples.** (a) Consider the "dishpan surface," the graph of the function f: $R^2\to R$ defined by

$$f(x, y) = \begin{cases} -\exp\left(\dfrac{+1}{x^2+y^2-r^2}\right), & x^2+y^2<r^2, \\ 0, & x^2+y^2\geqq r^2, \end{cases}$$

where $0\leqq r<\infty$. The index of nullity μ of the curvature operator of this surface is 0 on $G=\{(x, y)\,|\,x^2+y^2<r^2\}$ except for one circle and 2 on the complement of G. By choosing the function f differently the set G can be made to assume a variety of shapes; for example, polygonal regions or the complement of a finite set.

(b) The index of nullity of the Cartesian product of n dishpan surfaces assumes the values $0, 2, \cdots, 2n$.

(c) Let F be a flat manifold of dimension k: for example, R^k or a k-dimensional flat torus T^k. If the index of nullity of a manifold M assumes its minimum value λ on $G \subset M$, then the index of nullity of the curvature operator assumes its minimum value $\lambda + k$ on $G \times F$. If $\lambda = 0$, the folliation of $G \times F$ consists of manifolds of the form $\{p\} \times F$ ($p \in M$).

(d) Let M be an n-dimensional manifold and let $\mathcal{A}$ be $\mathcal{R}_K$ or $\mathcal{K}_K$, where $K \neq 0$. The author conjectures that the minimum value of the index of nullity of $\mathcal{A}$ is either 0 or n.

In the preparation of this paper the author has found conversations with B. O'Neill and R. Maltz helpful.

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