$E_1 - V$ is finite. But $\tau'$ is a $T_1$-topology so there is a $U^* \subseteq \tau'$ such that $\emptyset \neq U^* \subseteq U$ and $U^* \cap E_1 = \emptyset$. Thus $U^* \cap E_2 = U^*$ and $U^* \subseteq \tau \setminus \tau'$ but $U^* \notin \mathcal{C}$. Thus if $\tau \setminus \tau' = 1$ then $\tau \setminus \tau' \notin \mathcal{C}$. Hence it has been verified that

**Theorem 2.** The lattice $\Lambda$ of $T_1$-topologies on an infinite set $E$ is not complemented.

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**SPACES WITH ACYCLIC POINT COMPLEMENTS**

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1. **Introduction.** All homology groups will be singular homology with integer coefficients, reduced in dimension zero. If $0 \leq n \leq \infty$, a space $X$ is $n$-acyclic if $H_q(X) = 0$ for all integers $q \leq n$.

**Definition.** A Hausdorff space $M$ is an $A^n$-space if the complement of each point in $M$ is $n$-acyclic.

The condition on a point $x$ in $M$ that $M - x$ be $n$-acyclic is similar to the notion that $x$ be a non-$r$-cut point ($r \leq n$), defined by R. L. Wilder [9, p. 218], using Čech theory.

Clearly spheres are $A^\infty$-spaces. The object of this paper is to investigate to what extent $A^n$-spaces are like spheres. I wish to thank W. S. Massey for useful suggestions.

2. **Statement of results. Examples.** Open cells or closed cells of dimension $n + 2$ are clearly $A^n$-spaces. Hilbert space $l^2$ is an $A^\infty$-space; in fact by a theorem of Klee [5, p. 22], the complement of every compact subset of $l^2$ is homeomorphic to $l^2$ itself.

Received by the editors October 26, 1965.
A polyhedron $M$ is a homology $n$-manifold if and only if at all points $x$ of $M$ the local homology groups $H_q(M, M-x)$ are the same as those of Euclidean $n$-space. Every such space which is also a homology $n$-sphere must be an $A^n$-space. This may be proved directly by showing that for each $x$ in $M$ the homomorphism $H_*(M) \to H_*(M, M-x)$ in the exact sequence of the pair $(M, M-x)$ is an isomorphism. For examples of such spaces for $n \geq 3$ besides spheres one may take suspensions of Poincaré spaces [7, p. 218].

Let us now give an example, for each nonnegative integer $n$, of a compact $(n+1)$-dimensional polyhedron which is an $A^n$-space but which is neither a homology manifold nor a homology sphere. For this we need

**Theorem 1.** If $M$ is an $A^n$-space containing more than one point, then the suspension $S(M)$ of $M$ is an $A^{n+1}$-space.

Now let $\theta$ be a theta curve (a circle with a diameter). Clearly $\theta$ is an $A^0$-space. Thus by Theorem 1, the $n$-fold suspension $S(n)(\theta)$ of $\theta$ is an $A^n$-space. In dimension $n+1$, the homology group of $S(n)(\theta)$ and, at some points, the local homology group of $S(n)(\theta)$, are free abelian of rank 2.

In contrast to this example, the following theorem shows that things are nicer when the dimension of a polyhedral $A^n$-space is equal to $n$.

**Theorem 2.** If $M$ is an $n$-dimensional polyhedron which is an $A^n$-space, then $M$ is a homology manifold and a homology sphere.

For dimensions less than 3 we get the following characterization, which is similar to those in [9, pp. 220–223] using Čech theory.

**Theorem 3.** If $0 \leq n \leq 2$ and $M$ is an $A^n$-space having $H_n(M) \neq 0$, then $M$ is a topological $n$-sphere.

This follows from Theorem 2 in case $M$ is a polyhedron. However note that the only point-set assumption put on $M$ is that it be a Hausdorff space. This condition is crucial; for it is shown in [6] that for each $n \geq 0$ there exists a finite $T_0$ space (having $2n+2$ points) which has the same (singular) homology groups and homotopy groups as the $n$-sphere and in which the complement of each point is $\infty$-acyclic.

The sphere-like character of $A^n$-spaces is illustrated by the following theorem, which generalizes that part of the Alexander duality theorem dealing with the complement of a cell or of a sphere in a sphere.
Theorem 4. Let \(0 \leq k \leq n \leq \infty\) and let \(M\) be an \(A^n\)-space.

(a) If \(D\) is a closed \(k\)-cell in \(M\), then \(M - D\) is \((n-k)\)-acyclic.

(b) If \(S^k_0\) is a \(k\)-sphere in \(M\), then \(H_p(M - S^k_0) \approx H_{p+k+1}(M)\) whenever \(0 \leq p < n-k\).

The argument is analogous to the line of reasoning in [1] and also overlaps with the proof in [2] of the corresponding part of the Alexander duality theorem. Examples will be given to show that in part (a) we cannot assert that \(M-D\) is \((n-k+1)\)-acyclic and that in part (b) we do not obtain the isomorphism for \(p = n-k\).

3. Notation for the proofs. For each \(n \geq 0\) let \(R^n\) be Euclidean \(n\)-space, consisting of all infinite sequences \((x_1, x_2, \cdots)\) of reals such that \(x_i = 0\) for \(i > n\). (Thus \(R^n \subset R^{n+1}\).) In \(R^n\) let \(D^n\) be the unit disk \(\{x: \|x\| \leq 1\}\) and let \(S^{n-1}\) be its boundary. The standard \(n\)-cube \(I^n\) is the \(n\)-fold product of the unit interval \(I = [0, 1]\).

4. Proof of Theorem 1. Let \(\nu: M \times I \to S(M)\) be the quotient map, identifying \(M \times 0\) and \(M \times 1\) to the poles \(p_0\) and \(p_1\), respectively. The complement of either \(p_0\) or \(p_1\) is contractible. Hence consider the complement of \(y = \nu(x, t)\), where \(0 < t < 1\). In the proper triad

\[(S(M) - y; \nu(M \times [0, t]) - y, \nu(M \times [t, 1]) - y),\]

the second and third members are contractible, their union is the first member, and their intersection is homeomorphic to \(M - x\), which is nonempty. Hence the (reduced) Mayer-Vietoris sequence of this triad gives the isomorphism \(H_q(M - x) \approx H_{q+1}(S(M) - y)\), all \(q\).

5. Proof of Theorem 2. For each point \(x\) of \(M\), the group \(H_q(M - x)\) is zero for \(q \leq n\) since \(M\) is an \(A^n\)-space, and is zero for \(q > n\) since \(M\) is an \(n\)-dimensional polyhedron. Hence from the exact sequence of the pair \((M, M - x)\), we see that \(H_*(M, M - x) \approx H_*(M)\). On the other hand, choose some point \(x_0\) in a principal simplex of \(M\), so that \(x_0\) has a neighborhood \(V \approx R^n\). Thus \(H_*(M) \approx H_*(M, M - x_0) \approx H_*(V, V - x_0) \approx H_*(R^n, R^n - 0) \approx H_*(S^n)\).

6. Proof of Theorem 4. (a) The proof is by induction on the dimension \(k\) of \(D\), the case \(k = 0\) holding since \(M\) is an \(A^n\)-space. Suppose that \(k\) is a positive integer \(\leq n\) and that the result holds for \((k-1)\)-cells. Let \(D\) be the image of an imbedding \(h\) of \(I^k\) into \(M\). Suppose \(0 \leq k \leq n - k\) and suppose there exists a nonbounding \(q\)-cycle \(z\) in \(M - D\). Bisect \(D\) as follows. Consider the \(k\)-cells \(D_1 = h([0, 1/2] \times I^{k-1}), D_2 = h([1/2, 1] \times I^{k-1})\), and the \((k-1)\)-cell \(D_0 = h(\{1/2\} \times I^{k-1})\). Since \(q + 1 \leq n - (k-1)\), the inductive assumption and the
Mayer-Vietoris sequence of the proper triad \((M - D_0; M - D_1, M - D_2)\) show us that either \(z \sim 0\) in \(M - D_1\) or \(z \sim 0\) in \(M - D_2\). (This triad is proper because its members are open subsets of \(M\); see [3, p. 34, p. 199].) A continuation of the bisection process and the direct limit theorem for singular homology show that \(z \sim 0\) in the complement of a \((k-1)\)-cell, which is a contradiction.

(b) For each \(i \geq 0\), write the standard \(i\)-sphere \(S^i\) as the union of its upper cap \(U^i\) and its lower cap \(L^i\), so that \(U^i \cap L^i = S^{i-1}\). Now let \(S_0^k\) be the image of an imbedding \(h\) of \(S^k\) into \(M\). For \(i \leq k\), let \(S^i_0 = h(S^i)\), \(U^i_0 = h(U^i)\), and \(L^i_0 = h(L^i)\). Then part (a) and the Mayer-Vietoris sequence of the proper triad \((M - S^{i-1}_0; M - U^i_0, M - L^i_0)\) give the isomorphism

\[
H_q(M - S^{i-1}_0) \cong H_{q+1}(M - S^i_0) \quad (0 \leq i \leq k, 0 \leq q < n - i).
\]

Applying this successively, we get the chain of isomorphisms

\[
H_p(M - S^{k}_0) \cong H_{p+1}(M - S^{k-1}_0) \cong \cdots \cong H_{p+k}(M - S^{0}_0) \\
\cong H_{p+k+1}(M - S^{-1}_0) = H_{p+k+1}(M).
\]

This completes the proof of Theorem 4.

Now let us give the examples promised after the statement of the theorem. The complement of the \(k\)-cell \(D^k\) in the \(A^n\)-space \(D^{n+2}\) has the same homotopy type as \(S^{n-k+1}\), hence is not \((n-k+1)\)-acyclic. It is easy to see from the proof of part (b) that in the critical case \(p = n - k\) we always get \(H_{n-k}(M - S^0)\) as a homomorphic image of \(H_{n+1}(M)\). To get an example where \(H_{n-k}(M - S^0) = 0\) but \(H_{n+1}(M) \neq 0\), we may take \(M\) to be the \(A^n\)-space \(R^{n+2} - 0\) with \(S^k_0 = S^k\).

7. Proof of Theorem 3. The case \(n = 0\) of the theorem is trivial. Suppose now that \(n\) is any positive integer and that \(M\) is an \(A^n\)-space with \(H_n(M) \neq 0\). Let us show that \(M\) is a Peano space (locally connected, compact, connected, metrizable space). Let \(\{x, y\} \subset M\). There must be a point \(w\) in \(M\) such that \(\{x, y\} \cup M - w\), since \(H_n(M) \neq 0\). Since \(H_0(M - w) = 0\), there is a path from \(x\) to \(y\). Thus \(M\) is pathwise connected. Now choose a nonbounding \(n\)-cycle \(z = \sum_{i=1}^r m_i T_i\) on \(M\) (\(m_i\) an integer, \(T_i\) a singular \(n\)-simplex). The carrier \(|z| = \bigcup_{i=1}^r \) must cover all of \(M\), since for each point \(x\) of \(M\), \(H_0(M - x) = 0\). Since \(M\) is pathwise connected, it is easy then to construct a map of a Peano space (consisting of \(r\) \(n\)-simplexes and \((r-1)\) arcs) onto \(M\). But every Hausdorff space which is the continuous image of a Peano space is itself a Peano space (see for instance [4, p. 126]).
Now let us show that every \((n-1)\)-sphere \(S^{n-1}_0\) in \(M\) separates \(M\). By part (b) of Theorem 4, \(H_0(M - S^{n-1}_0) \approx H_n(M) \neq 0\). Hence \(M - S^{n-1}_0\) has more than one path-component. But the path-components of an open subset of a Peano space are the same as its components \([4, p. 118]\). It can be shown using part (a) of Theorem 4 that \(S^{n-1}_0\) is the boundary of each of the components of \(M - S^{n-1}_0\), but this is not needed in the following.

Case \(n = 1\). By the preceding, \(M\) is a compact, connected metrizable space (containing more than one point, since \(H_1(M) \neq 0\)) which is separated by every pair of its points. Hence by \([4, p. 55]\) \(M\) is homeomorphic to \(S^1\).

Case \(n = 2\). Zippin's \([10]\) characterization of the 2-sphere implies that \(M\) is homeomorphic to \(S^2\) provided that (i) \(M\) is a Peano space, (ii) every simple closed curve in \(M\) separates \(M\), (iii) no arc in \(M\) separates \(M\), and (iv) \(M\) contains a simple closed curve. Conditions (i) and (ii) have been established. Condition (iii) follows from Theorem 4(a) with \(n = 2\) and \(k = 1\). If condition (iv) did not hold, then by condition (i), \(M\) would be (by definition) a dendrite \([8, p. 88]\). But a dendrite contains (uncountably many) cut points \([8, p. 88]\). This contradicts the fact that \(M\) is an \(A^2\)-space. The proof of Theorem 3 is complete.

References


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