$E_1 - V$ is finite. But τ' is a T_1-topology so there is a $U^* \in \tau'$ such that $\emptyset \neq U^* \subset U$ and $U^* \cap E_1 = \emptyset$. Thus $U^* \cap E_2 = U^*$ and $U^* \in \tau \wedge \tau'$ but $U^* \notin \mathcal{C}$. Thus if $\tau \vee \tau' = 1$ then $\tau \wedge \tau' \neq \mathcal{C}$. Hence it has been verified that

THEOREM 2. *The lattice Λ of T_1-topologies on an infinite set E is not complemented.*

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SPACES WITH ACYCLIC POINT COMPLEMENTS

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1. **Introduction.** All homology groups will be singular homology with integer coefficients, reduced in dimension zero. If $0 \leqq n \leqq \infty$, a space X is *n-acyclic* if $H_q(X) = 0$ for all integers $q \leqq n$.

DEFINITION. A Hausdorff space M is an *A^n-space* if the complement of each point in M is n-acyclic.

The condition on a point x in M that $M - x$ be n-acyclic is similar to the notion that x be a non-r-cut point ($r \leqq n$), defined by R. L. Wilder [**9**, p. 218], using Čech theory.

Clearly spheres are A^∞-spaces. The object of this paper is to investigate to what extent A^n-spaces are like spheres. I wish to thank W. S. Massey for useful suggestions.

2. **Statement of results. Examples.** Open cells or closed cells of dimension $n+2$ are clearly A^n-spaces. Hilbert space l^2 is an A^∞-space; in fact by a theorem of Klee [**5**, p. 22], the complement of every compact subset of l^2 is homeomorphic to l^2 itself.

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A polyhedron M is a *homology n-manifold* if and only if at all points x of M the local homology groups $H_q(M, M-x)$ are the same as those of Euclidean n-space. Every such space which is also a homology n-sphere must be an A^∞-space. This may be proved directly by showing that for each x in M the homomorphism $H_*(M) \to H_*(M, M-x)$ in the exact sequence of the pair $(M, M-x)$ is an isomorphism. For examples of such spaces for $n \geqq 3$ besides spheres one may take suspensions of Poincaré spaces [7, p. 218].

Let us now give an example, for each nonnegative integer n, of a *compact* $(n+1)$*-dimensional polyhedron which is an* A^n*-space but which is neither a homology manifold nor a homology sphere.* For this we need

THEOREM 1. *If M is an A^n-space containing more than one point, then the suspension $S(M)$ of M is an A^{n+1}-space.*

Now let θ be a theta curve (a circle with a diameter). Clearly θ is an A^0-space. Thus by Theorem 1, the n-fold suspension $S^{(n)}(\theta)$ of θ is an A^n-space. In dimension $n+1$, the homology group of $S^{(n)}(\theta)$ and, at some points, the local homology group of $S^{(n)}(\theta)$, are free abelian of rank 2.

In contrast to this example, the following theorem shows that things are nicer when the dimension of a polyhedral A^n-space is equal to n.

THEOREM 2. *If M is an n-dimensional polyhedron which is an A^n-space, then M is a homology manifold and a homology sphere.*

For dimensions less than 3 we get the following characterization, which is similar to those in [9, pp. 220–223] using Čech theory.

THEOREM 3. *If $0 \leqq n \leqq 2$ and M is an A^n-space having $H_n(M) \neq 0$, then M is a topological n-sphere.*

This follows from Theorem 2 in case M is a polyhedron. However note that the only point-set assumption put on M is that it be a Hausdorff space. This condition is crucial; for it is shown in [6] that for each $n \geqq 0$ there exists a *finite* T_0 space (having $2n+2$ points) which has the same (singular) homology groups and homotopy groups as the n-sphere and in which the complement of each point is ∞-acyclic.

The sphere-like character of A^n-spaces is illustrated by the following theorem, which generalizes that part of the Alexander duality theorem dealing with the complement of a cell or of a sphere in a sphere.

THEOREM 4. *Let* $0 \leqq k \leqq n \leqq \infty$ *and let* M *be an* A^n*-space.*

(a) *If* D *is a closed* k*-cell in* M, *then* $M-D$ *is* $(n-k)$*-acyclic.*

(b) *If* S_0^k *is a* k*-sphere in* M, *then* $H_p(M-S_0^k) \approx H_{p+k+1}(M)$ *whenever* $0 \leqq p < n-k$.

The argument is analogous to the line of reasoning in [1] and also overlaps with the proof in [2] of the corresponding part of the Alexander duality theorem. Examples will be given to show that in part (a) we cannot assert that $M-D$ is $(n-k+1)$-acyclic and that in part (b) we do not obtain the isomorphism for $p=n-k$.

3. **Notation for the proofs.** For each $n \geqq 0$ let R^n be Euclidean n-space, consisting of all infinite sequences $(x_1, x_2, \cdots)$ of reals such that $x_i = 0$ for $i > n$. (Thus $R^n \subset R^{n+1}$.) In R^n let D^n be the unit disk $\{x : \|x\| \leqq 1\}$ and let S^{n-1} be its boundary. The standard n-cube I^n is the n-fold product of the unit interval $I = [0, 1]$.

4. **Proof of Theorem 1.** Let $\nu : M \times I \to S(M)$ be the quotient map, identifying $M \times 0$ and $M \times 1$ to the poles p_0 and p_1, respectively. The complement of either p_0 or p_1 is contractible. Hence consider the complement of $y = \nu(x, t)$, where $0 < t < 1$. In the proper triad

$$(S(M) - y; \nu(M \times [0, t]) - y, \nu(M \times [t, 1]) - y),$$

the second and third members are contractible, their union is the first member, and their intersection is homeomorphic to $M - x$, which is nonempty. Hence the (reduced) Mayer-Vietoris sequence of this triad gives the isomorphism $H_q(M-x) \approx H_{q+1}(S(M) - y)$, all q.

5. **Proof of Theorem 2.** For each point x of M, the group $H_q(M-x)$ is zero for $q \leqq n$ since M is an A^n-space, and is zero for $q > n$ since M is an n-dimensional polyhedron. Hence from the exact sequence of the pair $(M, M-x)$, we see that $H_*(M, M-x) \approx H_*(M)$. On the other hand, choose some point x_0 in a principal simplex of M, so that x_0 has a neighborhood $V \equiv R^n$. Thus $H_*(M) \approx H_*(M, M-x_0) \approx H_*(V, V-x_0) \approx H_*(R^n, R^n - 0) \approx H_*(S^n)$.

6. **Proof of Theorem 4.** (a) The proof is by induction on the dimension k of D, the case $k=0$ holding since M is an A^n-space. Suppose that k is a positive integer $\leqq n$ and that the result holds for $(k-1)$-cells. Let D be the image of an imbedding h of I^k into M. Suppose $0 \leqq q \leqq n-k$ and suppose there exists a nonbounding q-cycle z in $M-D$. *Bisect* D as follows. Consider the k-cells $D_1 = h([0, 1/2] \times I^{k-1})$, $D_2 = h([1/2, 1] \times I^{k-1})$, and the $(k-1)$-cell $D_0 = h(\{1/2\} \times I^{k-1})$. Since $q+1 \leqq n-(k-1)$, the inductive assumption and the

Mayer-Vietoris sequence of the proper triad $(M-D_0;\, M-D_1,\, M-D_2)$ show us that either $z \not\sim 0$ in $M-D_1$ or $z \not\sim 0$ in $M-D_2$. (This triad is proper because its members are open subsets of M; see [**3**, p. 34, p. 199].) A continuation of the bisection process and the direct limit theorem for singular homology show that $z \not\sim 0$ in the complement of a $(k-1)$-cell, which is a contradiction.

(b) For each $i \geqq 0$, write the standard i-sphere S^i as the union of its upper cap U^i and its lower cap L^i, so that $U^i \cap L^i = S^{i-1}$. Now let S_0^k be the image of an imbedding h of S^k into M. For $i \leqq k$, let $S_0^i = h(S^i)$, $U_0^i = h(U^i)$, and $L_0^i = h(L^i)$. Then part (a) and the Mayer-Vietoris sequence of the proper triad $(M-S_0^{i-1};\, M-U_0^i,\, M-L_0^i)$ give the isomorphism

$$H_q(M - S_0^i) \approx H_{q+1}(M - S_0^{i-1}) \qquad (0 \leqq i \leqq k,\ 0 \leqq q < n - i).$$

Applying this successively, we get the chain of isomorphisms

$$\begin{aligned} H_p(M - S_0^k) &\approx H_{p+1}(M - S_0^{k-1}) \approx \cdots \approx H_{p+k}(M - S_0^0) \\ &\approx H_{p+k+1}(M - S_0^{-1}) = H_{p+k+1}(M). \end{aligned}$$

This completes the proof of Theorem 4.

Now let us give the examples promised after the statement of the theorem. The complement of the k-cell D^k in the A^n-space D^{n+2} has the same homotopy type as S^{n-k+1}, hence is not $(n-k+1)$-acyclic. It is easy to see from the proof of part (b) that in the critical case $p = n-k$ we always get $H_{n-k}(M-S_0^k)$ as a homomorphic image of $H_{n+1}(M)$. To get an example where $H_{n-k}(M-S_0^k)=0$ but $H_{n+1}(M) \neq 0$, we may take M to be the A^n-space $R^{n+2}-0$ with $S_0^k = S^k$.

7. **Proof of Theorem 3.** The case $n=0$ of the theorem is trivial. Suppose now that n is any positive integer and that M is an A^n-space with $H_n(M) \neq 0$. Let us show that M is a Peano space (locally connected, compact, connected, metrizable space). Let $\{x, y\} \subset M$. There must be a point w in M such that $\{x, y\} \subset M-w$, since $H_n(M) \neq 0$. Since $H_0(M-w)=0$, there is a path from x to y. Thus M is pathwise connected. Now choose a nonbounding n-cycle $z = \sum_{i=1}^r m_i T_i$ on M (m_i an integer, T_i a singular n-simplex). The carrier $|z| = \bigcup_{i=1}^r \operatorname{im} T_i$ must cover all of M, since for each point x of M, $H_n(M-x)=0$. Since M is pathwise connected, it is easy then to construct a map of a Peano space (consisting of r n-simplexes and $(r-1)$ arcs) *onto* M. But every Hausdorff space which is the continuous image of a Peano space is itself a Peano space (see for instance [**4**, p. 126]).

Now let us show that every $(n-1)$-sphere S_0^{n-1} in M separates M. By part (b) of Theorem 4, $H_0(M-S_0^{n-1}) \approx H_n(M) \neq 0$. Hence $M-S_0^{n-1}$ has more than one path-component. But the path-components of an open subset of a Peano space are the same as its components [**4**, p. 118]. It can be shown using part (a) of Theorem 4 that S_0^{n-1} is the boundary of each of the components of $M-S_0^{n-1}$, but this is not needed in the following.

Case $n=1$. By the preceding, M is a compact, connected metrizable space (containing more than one point, since $H_1(M) \neq 0$) which is separated by every pair of its points. Hence by [**4**, p. 55] M is homeomorphic to S^1.

Case $n=2$. Zippin's [**10**] characterization of the 2-sphere implies that M is homeomorphic to S^2 provided that (i) M is a Peano space, (ii) every simple closed curve in M separates M, (iii) no arc in M separates M, and (iv) M contains a simple closed curve. Conditions (i) and (ii) have been established. Condition (iii) follows from Theorem 4(a) with $n=2$ and $k=1$. If condition (iv) did not hold, then by condition (i), M would be (by definition) a dendrite [**8**, p. 88]. But a dendrite contains (uncountably many) cut points [**8**, p. 88]. This contradicts the fact that M is an A^2-space. The proof of Theorem 3 is complete.

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