


UNIVERSITY OF CALIFORNIA, RIVERSIDE

CORRECTION TO “ON MATRICES WHOSE REAL LINEAR COMBINATIONS ARE NONSINGULAR”

J. F. ADAMS, PETER D. LAX AND RALPH S. PHILLIPS

We are grateful to Professor B. Eckmann for pointing out an error in the proof of Lemma 4(b) of our paper [1]. This error invalidates Lemma 4(b) and that part of Theorem 1 which states the values of $Q(n)$, $Q_H(n)$. The error occurs immediately after the words “arguing as is usual for the complex case, we find”; it consists in manipulating as if the ground field $A$ were commutative.

The proof of Lemma 4(b) can be repaired, as will be shown below, but it leads to a different conclusion from that given. Our paper should therefore be corrected as follows.

(i) In Theorem 1, the values of $Q(n)$ and $Q_H(n)$ should read

"$Q(n) = \rho(\frac{1}{2}n) + 4, \quad Q_H(n) = \rho(\frac{1}{2}n) + 5."$

The two sentence paragraph following Theorem 1 should be deleted. It remains interesting to ask what topological phenomena (if any) can be related to our algebraic results.

(ii) In Lemma 4, part (b) should read

"$Q_H(n) + 3 \leq R(4n)."$

The proof is as follows.

Let $W$ be a $k$-dimensional space of $n \times n$ Hermitian matrices with entries from $Q$ which has the property $P$. The space $Q^n$ is a real vector space of dimension $4n$. For each $A \in W$ and each pure imaginary $\mu \in Q$ we consider the following real-linear transformation from $Q^n$ to itself:
We claim that the \((k+3)\)-dimensional space formed by such \(B\) has the property \(P\). For suppose that such a \(B\) is singular; then there is a nonzero \(x\) such that

\[
Ax = -x\mu;
\]

then we have

\[
x^*(Ax) = -x^*x\mu,
\]

\[
(x^*A)x = (x\mu)^*x = \mu x^*x.
\]

Since \(x^*x\) is real and nonzero, we have \(\mu = 0\); hence \(A\) is singular and \(A = 0\). This completes the proof.

(iii) In Lemma 5, there should be added a second part, reading

"(b) \(R_H(n) + 3 \leq Q(n)\)."

PROOF. Let \(W\) be a \(k\)-dimensional space of \(n \times n\) real symmetric matrices which has the property \(P\). Consider the matrices

\[
A + \mu I,
\]

where \(A\) runs over \(W\) and \(\mu\) runs over the pure imaginary elements of \(Q\). We claim that they form a space of dimension \(k+3\) with the property \(P\). In fact, suppose that such a matrix is singular; and suppose to begin with, that \(\mu\) is nonzero. Then the elements 1, \(\mu\) form an \(R\)-base for a subalgebra of \(Q\) which we may identify with \(C\). Choose a \(C\)-base of \(Q\); this splits \(Q^n\) as the direct sum of two copies of \(C^n\). Since the matrix \(A + \mu I\) acts on each summand, it must be singular on at least one. That is, the real symmetric matrix \(A\) has a nonzero complex eigenvalue which is purely imaginary, a contradiction. Hence \(\mu\) must be zero and \(B = A\). Now choose an \(R\)-base of \(Q\); this splits \(Q^n\) as the direct sum of 4 copies of \(R^n\). Since \(A\) acts on each summand, it must be singular on at least one. That is, \(A\) must be singular; hence \(A = 0\). This completes the proof.

(iv) The final paragraph of the paper should be deleted, and replaced by the following proof.

"Finally, Lemmas 5(b), 3 and 4(b) show that

\[
Q(n) - R_H(n) \geq 3,
\]

\[
Q_H(2n) - Q(n) \geq 1,
\]

\[
R(8n) - Q_H(2n) \geq 3.
\]

But we have already shown that
so all these inequalities are equalities. This completes the proof of Theorem 1."

We note that this method provides an alternative proof of Lemma 5 (\(R(8n) - R_B(n) \geq 7\)), without using the Cayley numbers.

**BIBLIOGRAPHY**


Manchester University, Manchester, England

New York University, and

Stanford University