Let \((x_m : m \in D)\) be a net, which we will call the \textit{iterate net}, in a topological space \(X\) such that, for each \(m \in D\), there is a net \((x^m_d : d \in D_m) \to x_m\). We will call the net
\[
\left( x^{\beta(m)}_\beta : \langle m, \beta \rangle \in D \times \prod_{m \in D} D_m \right),
\]
where the product set is directed by the product order, the \textit{composite net} of the system of nets. It is well known (see \([5, p. 69]\)) that if the iterate net converges then the composite net converges to the same limit. Indeed, this property helps characterize topologies through convergence classes (see \([5, p. 74]\)). We will show that the converse of this iterated limit theorem characterizes regular spaces.

\textbf{Theorem A.} A topological space is regular iff any iterate net converges to the limit of the composite net whenever that limit exists.

\textbf{Proof.} Let \(X\) be a regular topological space and suppose the composite net of a system of nets, with the above notation, converges to a point \(x \in X\). Let \(G\) be any open set containing \(x\). By regularity, we may choose an open set \(G^*\) such that \(x \in G^* \subseteq c(G^*) \subseteq G\). Now there exists an element \(\langle m^*, \beta^* \rangle\) in the product directed set such that \(x^m_{\beta(m)} \in G^*\) for all \(\langle m, \beta \rangle \geq \langle m^*, \beta^* \rangle\). Since we have \((x^m_d : d \in D_m) \to x_m\) and each \(x^m_{\beta(m)} \in G^*\) for \(m \geq m^*\), it follows that \(x_m \in c(G^*) \subseteq G\) for \(m \geq m^*\) and the limit of the iterate net is \(x\). Conversely, suppose \(X\) is not regular. Then there exists a point \(x\) and an open set \(G^*\) containing it such that \(c(G) \not\subseteq G^*\) for any open set \(G\) containing \(x\). Let \(\{G_m : m \in D\}\) be the family of all open sets containing \(x\), directed by inclusion. Since \(c(G_m) \not\subseteq G^*\) for each \(m \in D\), there exists a point \(x_m \in c(G_m) \setminus G^*\). Since \(x_m \in c(G_m)\), there exists a net \((x^m_d : d \in D_m)\) in \(G_m\) converging to \(x_m\). Since \(x_m \in G^*\) for each \(m \in D\), the iterate net \((x_m : m \in D)\) cannot converge to \(x\). The composite net does converge to \(x\), however, as the following shows. Let \(G\) be an arbitrary open set containing \(x\). Then \(G = G_d\) for some \(d \in D\). Let \(\langle d, \alpha \rangle\) be a member of the product directed set with \(\alpha\) fixed but arbitrary. Then if \(\langle m, \beta \rangle \geq \langle d, \alpha \rangle\), then \(x^{m}_{\beta(m)} \in G_m \subseteq G_d = G\). q.e.d.

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This characterization of regularity was motivated by results of Dieudonné [2] and of Cook and Fischer [3] who use a product of filters and a "compression" operator on families of filters, respectively.

Since the property of the theorem depends only on a notion of convergence for nets, we may call a poset regular with respect to its order convergence (or \( o \)-convergence as defined in [1, p. 60]) iff any iterate net converges to the limit of the composite net whenever that limit exists. We recall that a point \( x \) in a poset is a limit of a net \((x_m: m \in D)\) with respect to its order convergence iff \( \lim x_m = x \), where these limits are taken in the completion by cuts.

**Theorem B.** Every poset is regular in its order convergence.

**Proof.** In the above notation, we need only prove that \( \lim x_m \leq \lim \sup x_{\beta(m)} \) since then, by duality,

\[
x = \lim \inf x_{\beta(m)} \leq \lim \inf x_m \leq \lim \sup x_m \leq \lim \sup x_{\beta(m)} = x
\]

and so \((x_m: m \in D) \to x\) as desired. We must show, then, that

\[
\bigwedge_{m^*} \bigvee_{m \geq m^*} x_m \leq \bigwedge_{(m^*, \beta^*)} \bigvee_{(m^*, \beta) \geq (m^*, \beta^*)} x_{\beta(m)}.
\]

Let us fix \((m^*, \beta^*)\), which gives us an element

\[
\bigvee_{(m, \beta) \geq (m^*, \beta^*)} x_{\beta(m)}
\]

over which we take the infimum on the right-hand side, and we shall show that (for the same fixed \( m^* \)), the element

\[
\bigvee_{m \geq m^*} x_m
\]

is smaller, and so the infima are comparable as designated. We calculate that

\[
\bigvee_{m \geq m^*} x_m = \bigvee_{m \geq m^*} \bigwedge_{d \geq d^*} x_d \leq \bigwedge_{m \geq m^*} \bigvee_{d \geq \beta^*(m)} x_d \leq \bigwedge_{m \geq m^*} \bigvee_{d \geq d^*} x_d \leq \bigwedge_{(m, \beta) \geq (m^*, \beta^*)} \bigvee_{m \geq m^*} x_{\beta(m)},
\]

where the final inequality follows from the fact that for any element \( x_d \) on the left, we have \( m \geq m^* \) and \( d \geq \beta^*(m) \). Then by defining \( \beta(k) = d \) if \( k = m \) and \( = \beta^*(m) \) otherwise, we have \( x_{\beta(m)} = x_d \) with \( (m, \beta) \geq (m^*, \beta^*) \); thus the element appears on the right. q.e.d.
An application of this result is an immediate proof of a result of DeMarr [4]:

**COROLLARY.** Every O-space is a regular Hausdorff space.

**Proof.** The Hausdorff condition is immediate since limits are unique in a complete lattice. Suppose we have a system of nets with the composite net converging with respect to the topology. Since the space is an O-space, by definition it is homeomorphic to a subset of a complete lattice with each net converging with respect to the topology iff it o-converges to that limit. Since the complete lattice is regular by Theorem B, the iterate net o-converges to the same limit and hence also with respect to the topology. By Theorem A, the space is regular.

**BIBLIOGRAPHY**


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