COMPLETENESS, THE RECURSION THEOREM, AND EFFECTIVELY SIMPLE SETS

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1. **Introduction.** The main result of this note is that every effectively simple set is complete, i.e., of the highest recursively enumerable degree of unsolvability. A recursively enumerable set of natural numbers is called *simple* if its complement, though infinite, possesses no infinite recursive subset. Let $W_0, W_1, \cdots$ be the standard enumeration of all recursively enumerable (*r.e.*) sets. Smullyan [8] calls an r.e. set S *effectively simple* if $\overline{S}$ (the complement of S) is infinite and there is a recursive function f such that, for every number e, if $W_e \subseteq \overline{S}$ then $f(e)$ is greater than the cardinality of W_e. It is immediate that every effectively simple set is simple. In [6] Sacks shows by direct construction that not every simple set is effectively simple. Sacks' theorem is a consequence of our main result and Friedberg's solution of Post's problem since [1] every nonzero r.e. degree of unsolvability is the degree of a simple set.

McLaughlin [2] calls an r.e. set with an infinite complement (a *coinfinite* r.e. set) S strongly effectively simple if there is a recursive function f such that, for each e, if $W_e \subseteq \overline{S}$ then $f(e)$ is greater than every member of W_e. Strongly effectively simple sets are all effectively simple, but we can, by varying slightly Sacks' construction and argument [6], produce an effectively simple set which is not strongly effectively simple. McLaughlin [2] proves that every strongly effectively simple set is either hypersimple or complete, a fact which is subsumed under our result.

The completeness of effectively simple sets can be proved by a simple argument using the recursion theorem, but we prefer to attack a more general problem. Several kinds of r.e. sets (e.g., creative sets [4] and quasicreative sets [7]) have been proved complete by methods which involve the recursion theorem. Our aim is to capture the essence of these methods. To do this, we shall define a rather general class $\mathfrak{D}$ of r.e. sets. It will be almost trivial to show—directly from definitions—that such classes as those of creative sets, quasicreative sets, and effectively simple sets are included in $\mathfrak{D}$. We shall use the recursion theorem to show that every member of $\mathfrak{D}$ is complete. We shall conclude by showing that every truth-table complete set belongs to $\mathfrak{D}$.[1]

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[1] A. H. Lachlan has pointed out to the author that, by replacing h with a partial recursive functional in an appropriate manner, we could make $\mathfrak{D}$ into the class of all complete r.e. sets.

2. **The class $\mathfrak{D}$.** Let $\beta_0, \beta_1, \cdots$ be the standard enumeration of all partial recursive functions (of one argument). We define a partial recursive function $\pi(e, z)$ as follows:

$$\pi(e, z) = \mu y[(x)_{x \leq y}(\beta_e(x) \text{ is defined}) \text{ and there are at least } z \text{ numbers } x \leqq y \text{ for which } \beta_e(x) = 1].$$

The *representing function* of a set of numbers A is that function which is zero for all $x \in A$ and 1 for all $x \notin A$. We call a relation $P(x, A, B)$ between numbers x, sets A, and sets B *admissible* if there are recursive functions $g(e, x)$ and $h(x)$ with the following property: For all numbers e and x and for every set C, if $\pi(e, h(x))$ is defined and if the representing function of C agrees with β_e on all numbers $\leqq \pi(e, h(x))$, then

$$\sim P(x, W_{g(e,x)}, C),$$

where "$\sim$" means *not*.

The property of admissible relations $P(x, A, B)$ which is crucial for the theorem below is, in intuitive terms, this: Given a number x and a coinfinite recursive set C, we can effectively find the index of an r.e. set W such that $\sim P(x, W, C)$, and furthermore we have an effective bound $h(x)$ to the information about C which we will need to enumerate W.

We say that a relation $P(x, A, B)$ *is satisfied by* a set C if

$$(n)P(n, W_n, C).$$

We are now ready to define $\mathfrak{D}$. $\mathfrak{D}$ is to consist of those coinfinite r.e. sets D such that some admissible relation is satisfied by D.

THEOREM. *Every member of $\mathfrak{D}$ is complete.*

PROOF. Let D belong to $\mathfrak{D}$. Let $P(x, A, B)$ be an admissible relation satisfied by D, and let $g(e, x)$ and $h(x)$ be the functions whose existence is guaranteed by the admissibility of P. Let d be a recursive function whose range is D. For each s, we set $D^s = \{x: (Ey)_{y \leq s} d(y) = x\}$. For each s and x, let K_x^s be the set of the least x members of $\overline{D^s}$.

There is a recursive function $p(e, t)$ such that, for each e and t,

$$\beta_{p(e,t)} = \begin{cases} \text{the representing function of } D^{\beta_e(t)} \text{ if } \beta_e(t) \text{ is defined;} \\ \text{the empty partial function otherwise.} \end{cases}$$

By the recursion theorem, there is a recursive function $q(e, t)$ such that

$$(e)(t)[W_{g(p(e,t),\, q(e,t))} = W_{q(e,t)}].$$

For each e and t, let

$$v(e, t) = \mu s[K^s_{h(q(e,t))} \subseteq \overline{D}].$$

v is recursive in D.

Let W be any r.e. set. Let $f(s, x)$ be a recursive function such that

$$(x)[x \in W \leftrightarrow (Es)f(s, x) = 0].$$

Let e be an index of the partial recursive function $\mu s[f(s, x) = 0]$. We shall show that $\beta_e(t) < v(e, t)$ for each t such that $\beta_e(t)$ is defined. This will mean that W is recursive in v and hence in D, since we shall have

$$(x)[x \in W \leftrightarrow (Es)_{s<v(e,x)}f(s, x) = 0].$$

Suppose that, contrary to what we wish to establish, $\beta_e(t)$ is defined for some t and $\beta_e(t) \geqq v(e, t)$. Then, by the definitions of v and of K^s_x,

$$K^{\beta_e(t)}_{h(q(e,t))} \subseteq \overline{D}.$$

But this just says that the least $h(q(e, t))$ members of $(D^{\beta_e(t)})^-$ are in $\overline{D}$. Hence the representing function of D agrees with $\beta_{p(e,t)}$, the representing function of $D^{\beta_e(t)}$, on all numbers $\leqq \pi(p(e, t), h(q(e, t)))$. But then

$$\sim P(q(e, t), W_{g(p(e,t),q(e,t))}, D),$$

and the definition of $q(e, t)$ gives us that

$$\sim P(q(e, t), W_{q(e,t)}, D),$$

which contradicts the hypothesis that P is satisfied by D.

3. **Effectively simple sets.** We now show that every effectively simple set belongs to $\mathfrak{D}$. Let S be effectively simple and let f be a recursive function such that, for each e, if $W_e \subseteq \overline{S}$ then $f(e)$ is greater than the cardinality of W_e. Let $h(x) = f(x)$, for all x. Let $g(e, x)$ be a recursive function satisfying

$$y \in W_{g(e,x)} \leftrightarrow \pi(e, h(x)) \text{ is defined and } y \leqq \pi(e, h(x)) \text{ and } \beta_e(y) = 1.$$

Let $P(x, A, B)$ hold if and only if $h(x)$ is greater than the cardinality of A or $A \not\subseteq \overline{B}$.

Evidently P is satisfied by S. Using h and g, we can see that P is admissible, as follows:

Let x and e be such that $\pi(e, h(x))$ is defined and let C be a set whose representing function agrees with β_e on all numbers $\leqq \pi(e, h(x))$.

Since $\pi(e, h(x))$ is defined, $W_{g(e,x)}$ is the set of all numbers $\leqq \pi(e, h(x))$ which belong to $\overline{C}$. Hence $W_{g(e,x)} \subseteq \overline{C}$. Furthermore, $W_{g(e,x)}$ has $h(x) = f(x)$ members, so that $\sim P(x, W_{g(e,x)}, C)$.

In [8] Smullyan proves a theorem one of whose main consequences is the existence of hypersimple effectively simple sets. As a final remark on effectively simple sets, we point out that this consequence follows from general elementary facts: Evidently every coinfinite r.e. superset of an effectively simple set is effectively simple; and in [3] it is shown that every simple set has a hypersimple superset. Similar considerations show that there are hypersimple strongly effectively simple sets.

4. **Truth-table complete sets.** The truth-table complete sets were defined by Post [5] and are a proper subclass of the complete sets. For any function v, the function $\bar{v}$ is defined by $\bar{v}(x) = \prod_{i<x} p_i^{v(i)}$, where p_i is the $i+1$st prime number. An r.e. set V is *truth-table complete* if, for every r.e. set W, there are recursive functions $p(x, y)$ and $q(x)$ such that, for each x, $p(x, \bar{v}(q(x)))$ is the representing function of W, where v is the representing function of V.

THEOREM. *Every truth-table complete set is a member of* $\mathfrak{D}$.

PROOF. Let V be truth-table complete, and let v be its representing function. Let $W = \{x : x \in W_x\}$. Let p and q be recursive functions related to V and W as in the definition of truth-table completeness. Let

$$P(x, A, B) \leftrightarrow [x \in A \leftrightarrow p(x, \bar{b}(q(x))) = 0], \text{ where } b \text{ is the representing function of } B;$$

$$h(x) = q(x);$$

$$y \in W_{g(e,x)} \leftrightarrow p(y, \tilde{\beta}_e(q(y))) > 0,$$

with g recursive. Clearly P is satisfied by V. The reader may easily verify the admissibility of P, using h and g.

It is worth noting that not every member of $\mathfrak{D}$ is truth-table complete. In particular, Post [5] shows that no hypersimple set is truth-table complete, whereas we have seen in §3 that there are hypersimple sets which belong to $\mathfrak{D}$.

REFERENCES

1. J. C. E. Dekker, *A theorem on hypersimple sets*, Proc. Amer. Math. Soc. **5** (1954), 791–796.

2. T. G. McLaughlin, *On a class of complete sets*, Canad. Math. Bull. **8** (1965), 33–37.

3. D. A. Martin, *A theorem on hyperhypersimple sets*, J. Symbolic Logic **28** (1963), 273–278.

4. J. Myhill, *Creative sets*, Z. Math. Logik. Grundlagen Math. **1** (1955), 97–108.

5. E. Post, *Recursively enumerable sets of postivie integers and their decision problems*, Bull. Amer. Math. Soc. **50** (1944), 284–316.

6. G. E. Sacks, *A simple set which is not effectively simple*, Proc. Amer. Math. Soc. **15** (1964), 51–55.

7. J. R. Schoenfield, *Quasicreative sets*, Proc. Amer. Math. Soc. **8** (1957), 964–967.

8. R. M. Smullyan, *Effectively simple sets*, Proc. Amer. Math. Soc. **15** (1964), 893–894.

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