COMPLETENESS, THE RECURSION THEOREM, AND EFFECTIVELY SIMPLE SETS

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1. Introduction. The main result of this note is that every effectively simple set is complete, i.e., of the highest recursively enumerable degree of unsolvability. A recursively enumerable set of natural numbers is called simple if its complement, though infinite, possesses no infinite recursive subset. Let \( W_0, W_1, \cdots \) be the standard enumeration of all recursively enumerable (r.e.) sets. Smullyan [8] calls an r.e. set \( S \) effectively simple if \( \overline{S} \) (the complement of \( S \)) is infinite and there is a recursive function \( f \) such that, for every number \( e \), if \( W_e \subseteq \overline{S} \) then \( f(e) \) is greater than the cardinality of \( W_e \). It is immediate that every effectively simple set is simple. In [6] Sacks shows by direct construction that not every simple set is effectively simple. Sacks' theorem is a consequence of our main result and Friedberg's solution of Post's problem since [1] every nonzero r.e. degree of unsolvability is the degree of a simple set.

McLaughlin [2] calls an r.e. set with an infinite complement (a coinfinite r.e. set) \( S \) strongly effectively simple if there is a recursive function \( f \) such that, for each \( e \), if \( W_e \subseteq \overline{S} \) then \( f(e) \) is greater than every member of \( W_e \). Strongly effectively simple sets are all effectively simple, but we can, by varying slightly Sacks' construction and argument [6], produce an effectively simple set which is not strongly effectively simple. McLaughlin [2] proves that every strongly effectively simple set is either hypersimple or complete, a fact which is subsumed under our result.

The completeness of effectively simple sets can be proved by a simple argument using the recursion theorem, but we prefer to attack a more general problem. Several kinds of r.e. sets (e.g., creative sets [4] and quasicreative sets [7]) have been proved complete by methods which involve the recursion theorem. Our aim is to capture the essence of these methods. To do this, we shall define a rather general class \( \mathcal{D} \) of r.e. sets. It will be almost trivial to show—directly from definitions—that such classes as those of creative sets, quasicreative sets, and effectively simple sets are included in \( \mathcal{D} \). We shall use the recursion theorem to show that every member of \( \mathcal{D} \) is complete. We shall conclude by showing that every truth-table complete set belongs to \( \mathcal{D} \).

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1 A. H. Lachlan has pointed out to the author that, by replacing \( h \) with a partial recursive functional in an appropriate manner, we could make \( \mathcal{D} \) into the class of all complete r.e. sets.
2. **The class** \( \mathcal{D} \). Let \( \beta_0, \beta_1, \cdots \) be the standard enumeration of all partial recursive functions (of one argument). We define a partial recursive function \( \pi(e, z) \) as follows:

\[
\pi(e, z) = \mu y \left[ (x) z_x y (\beta_x(x) \text{ is defined}) \text{ and there are at least } z \text{ numbers } x \leq y \text{ for which } \beta_x(x) = 1 \right].
\]

The *representing function* of a set of numbers \( A \) is that function which is zero for all \( x \in A \) and 1 for all \( x \notin A \). We call a relation \( P(x, A, B) \) between numbers \( x \), sets \( A \), and sets \( B \) *admissible* if there are recursive functions \( g(e, x) \) and \( h(x) \) with the following property: For all numbers \( e \) and \( x \) and for every set \( C \), if \( \pi(e, h(x)) \) is defined and if the representing function of \( C \) agrees with \( P \) on all numbers \( \leq \pi(e, h(x)) \), then

\[
\sim P(x, W_{g(e,x)}, C),
\]

where \( \sim \) means *not*.

The property of admissible relations \( P(x, A, B) \) which is crucial for the theorem below is, in intuitive terms, this: Given a number \( x \) and a coinfinte recursive set \( C \), we can effectively find the index of an r.e. set \( W \) such that \( \sim P(x, W, C) \), and furthermore we have an effective bound \( h(x) \) to the information about \( C \) which we will need to enumerate \( W \).

We say that a relation \( P(x, A, B) \) is *satisfied* by a set \( C \) if

\[
(n) P(n, W_n, C).
\]

We are now ready to define \( \mathcal{D} \). \( \mathcal{D} \) is to consist of those coinfinte r.e. sets \( D \) such that some admissible relation is satisfied by \( D \).

**Theorem.** *Every member of \( \mathcal{D} \) is complete.*

**Proof.** Let \( D \) belong to \( \mathcal{D} \). Let \( P(x, A, B) \) be an admissible relation satisfied by \( D \), and let \( g(e, x) \) and \( h(x) \) be the functions whose existence is guaranteed by the admissibility of \( P \). Let \( d \) be a recursive function whose range is \( D \). For each \( s \), we set \( D^s = \{ x: (E y)_{y \leq s} d(y) = x \} \). For each \( s \) and \( x \), let \( K^s_x \) be the set of the least \( x \) members of \( D^s \).

There is a recursive function \( p(e, t) \) such that, for each \( e \) and \( t \),

\[
\beta_{p(e,t)} = \begin{cases} 
\text{the representing function of } D^{\beta_e(t)} \text{ if } \beta_e(t) \text{ is defined}; \\
\text{the empty partial function otherwise.} 
\end{cases}
\]

By the recursion theorem, there is a recursive function \( q(e, t) \) such that

\[
(q)(t)[W_{g(p(e,t), q(e,t))} = W_{q(e,t)}].
\]
For each \( e \) and \( t \), let
\[
v(e, t) = \mu s[K_{h(q(e, t))}^s \subseteq \overline{D}].
\]
v is recursive in \( D \).

Let \( W \) be any r.e. set. Let \( f(s, x) \) be a recursive function such that
\[
(x)[x \in W \iff (Es) f(s, x) = 0].
\]
Let \( e \) be an index of the partial recursive function \( \mu s[f(s, x) = 0] \). We shall show that \( \beta_e(t) < v(e, t) \) for each \( t \) such that \( \beta_e(t) \) is defined. This will mean that \( W \) is recursive in \( v \) and hence in \( D \), since we shall have
\[
(x)[x \in W \iff (Es) f(s, x) = 0].
\]
Suppose that, contrary to what we wish to establish, \( \beta_e(t) \) is defined for some \( t \) and \( \beta_e(t) \geq v(e, t) \). Then, by the definitions of \( v \) and of \( K_{e, t} \),
\[
K_{h(q(e, t))}^{\beta_e(t)} \subseteq \overline{D}.
\]
But this just says that the least \( h(q(e, t)) \) members of \( (D^{\beta_e(t)})^- \) are in \( \overline{D} \). Hence the representing function of \( D \) agrees with \( \beta_{p(e, t)} \), the representing function of \( D^{\beta_e(t)} \), on all numbers \( \leq \pi(p(e, t), h(q(e, t))) \). But then
\[
\sim P(q(e, t), W_{q(p(e, t), q(e, t))}, D),
\]
and the definition of \( q(e, t) \) gives us that
\[
\sim P(q(e, t), W_{q(e, t)}, D),
\]
which contradicts the hypothesis that \( P \) is satisfied by \( D \).

3. **Effectively simple sets.** We now show that every effectively simple set belongs to \( \mathfrak{D} \). Let \( S \) be effectively simple and let \( f \) be a recursive function such that, for each \( e \), if \( W_e \subseteq S \) then \( f(e) \) is greater than the cardinality of \( W_e \). Let \( h(x) = f(x) \), for all \( x \). Let \( g(e, x) \) be a recursive function satisfying
\[
y \in W_{q(e, x)} \iff \pi(e, h(x)) \text{ is defined and } y \leq \pi(e, h(x)) \text{ and } \beta_e(y) = 1.
\]
Let \( P(x, A, B) \) hold if and only if \( h(x) \) is greater than the cardinality of \( A \) or \( A \subseteq \overline{B} \).

Evidently \( P \) is satisfied by \( S \). Using \( h \) and \( g \), we can see that \( P \) is admissible, as follows:

Let \( x \) and \( e \) be such that \( \pi(e, h(x)) \) is defined and let \( C \) be a set whose representing function agrees with \( \beta_e \) on all numbers \( \leq \pi(e, h(x)) \).
Since \( \pi(e, h(x)) \) is defined, \( W_{\varphi(e, x)} \) is the set of all numbers \( \leq \pi(e, h(x)) \) which belong to \( \mathcal{C} \). Hence \( W_{\varphi(e, x)} \subseteq \mathcal{C} \). Furthermore, \( W_{\varphi(e, x)} \) has \( h(x) = f(x) \) members, so that \( \sim P(x, W_{\varphi(e, x)}, C) \).

In [8] Smullyan proves a theorem one of whose main consequences is the existence of hypersimple effectively simple sets. As a final remark on effectively simple sets, we point out that this consequence follows from general elementary facts: Evidently every coinfinite r.e. superset of an effectively simple set is effectively simple; and in [3] it is shown that every simple set has a hypersimple superset. Similar considerations show that there are hypersimple strongly effectively simple sets.

4. Truth-table complete sets. The truth-table complete sets were defined by Post [5] and are a proper subclass of the complete sets. For any function \( v \), the function \( \check{v} \) is defined by \( \check{v}(x) = \prod_{i \leq x} p_i^{v(i)} \), where \( p_i \) is the \( i+1 \)st prime number. An r.e. set \( V \) is truth-table complete if, for every r.e. set \( W \), there are recursive functions \( p(x, y) \) and \( q(x) \) such that, for each \( x \), \( p(x, \check{v}(q(x))) \) is the representing function of \( W \), where \( v \) is the representing function of \( V \).

**Theorem.** Every truth-table complete set is a member of \( \mathcal{D} \).

**Proof.** Let \( V \) be truth-table complete, and let \( v \) be its representing function. Let \( W = \{ x : x \in W_x \} \). Let \( p \) and \( q \) be recursive functions related to \( V \) and \( W \) as in the definition of truth-table completeness. Let

\[
P(x, A, B) \iff \{ x \in A \iff p(x, \check{v}(q(x))) = 0 \}, \text{ where } b \text{ is the representing function of } B;
\]

\[
h(x) = q(x);
\]

\[
y \in W_{\varphi(e, x)} \iff p(y, \check{v}(q(y))) > 0,
\]

with \( g \) recursive. Clearly \( P \) is satisfied by \( V \). The reader may easily verify the admissibility of \( P \), using \( h \) and \( g \).

It is worth noting that not every member of \( \mathcal{D} \) is truth-table complete. In particular, Post [5] shows that no hypersimple set is truth-table complete, whereas we have seen in §3 that there are hypersimple sets which belong to \( \mathcal{D} \).

**References**


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