LINEAR MAPPINGS OF OPERATOR ALGEBRAS

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In (6) it was shown that a linear mapping \( \phi \) of one C*-algebra \( \mathfrak{a} \) with identity into another which carries unitary operators into unitary operators is a C*-homomorphism followed by multiplication by the unitary operator \( \phi(I) \), i.e. \( \phi(A) = \phi(I)\rho(A) \), \( \phi(A^*) = \phi(A)^* \), and \( \rho(A^2) = \rho(A)^2 \) for each \( A \) in \( \mathfrak{a} \). We continue in that spirit here, with the unitary group replaced first by an arbitrary semigroup contained in the unit sphere, then by the semigroup of regular contractions. By a C*-algebra we shall mean a uniformly closed self-adjoint algebra of bounded linear operators on some Hilbert space, which contains the identity operator.

**Lemma 1.** Let \( \mathfrak{B} \) be a normed algebra containing a multiplicative semigroup \( \mathfrak{s} \) with the following properties: (i) the linear span of \( \mathfrak{s} \) is \( \mathfrak{B} \); (ii) \( \sup \{ \| s \| : s \in \mathfrak{s} \} = K < \infty \). For \( x \) in \( \mathfrak{B} \), define \( \| x \|_\mathfrak{s} \) to be \( \inf \{ \sum_1^n | a_j | : x = \sum_1^n a_j s_j, s_j \in \mathfrak{s}, a_j \text{ complex}, n \geq 1 \} \). Then \( \| \cdot \|_\mathfrak{s} \) is a normed algebra norm on \( \mathfrak{B} \) such that \( \| \cdot \|_\mathfrak{s} \leq K \| \cdot \|_\mathfrak{B} \). Furthermore, if \( \mathfrak{s} \) and \( \mathfrak{z} \) are multiplicative semigroups in the normed algebras \( \mathfrak{B} \) and \( \mathfrak{c} \) resp., each satisfying (i) and (ii), and if \( \phi \) is a linear mapping of \( \mathfrak{B} \) into \( \mathfrak{c} \) such that \( \phi(\mathfrak{s}) \subseteq \mathfrak{z} \), then for each \( x \) in \( \mathfrak{B} \), \( \| \phi(x) \|_\mathfrak{z} \leq \| x \|_\mathfrak{s} \).

**Proof.** Verify.

Let \( \mathfrak{a} \) be a C*-algebra and let \( \mathfrak{s} \) be a multiplicative semigroup contained in the unit sphere of \( \mathfrak{a} \). Suppose that the linear span of \( \mathfrak{s} \) is \( \mathfrak{a} \) and that \( \| A \|_\mathfrak{s} = \| A \| \) whenever \( A \) is a regular element of \( \mathfrak{a} \). For example \( \mathfrak{s} \) could be the group of unitary operators, the semigroup of regular contractions, or the entire unit sphere of \( \mathfrak{a} \).

**Lemma 2.** Let \( \phi \) be a linear mapping of \( \mathfrak{a} \) into a C*-algebra \( \mathfrak{B} \) such that \( \phi(I) = I \) and \( \phi \) maps \( \mathfrak{s} \) into the unit sphere of \( \mathfrak{B} \). Then \( \phi \) is a self-adjoint mapping, i.e. \( \phi(A^*) = \phi(A)^* \).

**Proof.** We argue as in [2, Lemma 8]. Let \( A \) be a self-adjoint element of \( \mathfrak{a} \) of norm 1. Then \( \phi(A) = B + iC \), where \( B \) and \( C \) are self-adjoint elements of \( \mathfrak{B} \). If \( C \neq 0 \), let \( b \) be a positive number in the spectrum of \( C \) (otherwise consider \(-C\)). Choose a positive integer \( n \) such that \( (1+n^2)^{1/2} < b + n \). Then since \( A + inI \) is regular, \( \| A + inI \| = (1 + n^2)^{1/2} < b + n \leq \| iC + inI \| \leq \| B + i(C + nI) \| = \| \phi(A + inI) \| \).
\[\|A + inI\|_S = \|A + inI\|,\] a contradiction. It follows that \(\phi\) is a self-adjoint mapping.

**Theorem 1.** Let \(\mathfrak{A}\) and \(\mathfrak{B}\) be \(C^*\)-algebras and let \(S(\text{resp. } S)\) be a multiplicative semigroup contained in the unit sphere of \(\mathfrak{A}\) (resp. \(\mathfrak{B}\)). Suppose that \(\|A\|_S = \|A\|\) (resp. \(\|B\|_S = \|B\|\)) whenever \(A\) (resp. \(B\)) is a regular element of \(\mathfrak{A}\) (resp. \(\mathfrak{B}\)). Let \(\phi\) be a one-to-one linear mapping of \(\mathfrak{A}\) onto \(\mathfrak{B}\) such that \(\phi(I) = I\), \(\phi\) maps \(S\) into the unit sphere of \(\mathfrak{B}\), and \(\phi^{-1}\) maps \(S\) into the unit sphere of \(\mathfrak{A}\). Then \(\phi\) is a \(C^*\)-isomorphism.

**Proof.** By Lemma 2, \(\phi\) is a self-adjoint mapping. If \(A\) is a self-adjoint element of \(\mathfrak{A}\) then \(A + iI\) is regular and \((\|\phi(A)\|_S^2 + 1)^{1/2} = \|\phi(A + iI)\|_S = \|A + iI\|_S = (\|A\|_S^2 + 1)^{1/2}\), so that \(\|\phi(A)\|_S \leq \|A\|_S\). Similarly \(\|\phi^{-1}(B)\|_S \leq \|B\|_S\) for each self-adjoint element \(B\) of \(\mathfrak{B}\). Thus \(\phi\) is an isometry of the Jordan algebra of self-adjoint elements of \(\mathfrak{A}\) onto the Jordan algebra of self-adjoint elements of \(\mathfrak{B}\) [3]. By a theorem of Kadison [3, Theorem 2], \(\phi\) is a \(C^*\)-isomorphism.

The theorem shows that isometries of \(C^*\)-algebras which preserve the identity are \(C^*\)-isomorphisms [2, Theorem 7].

We now consider the semigroup \(R_1(\mathfrak{A})\) of all regular contractions of a \(C^*\)-algebra \(\mathfrak{A}\), i.e. the set of all invertible elements of \(\mathfrak{A}\) of norm at most one. Let \(\phi\) be a linear mapping of \(\mathfrak{A}\) into a \(C^*\)-algebra \(\mathfrak{B}\) such that \(\phi(I) = I\) and \(\phi(R_1(\mathfrak{A})) \subseteq R_1(\mathfrak{B})\). By Lemma 2, \(\phi\) is a self-adjoint mapping and clearly \(\phi(R(\mathfrak{A})) \subseteq R(\mathfrak{B})\), where \(R(\mathfrak{A})\) denotes the group of all regular elements of the \(C^*\)-algebra \(\mathfrak{A}\).

In case \(\mathfrak{A} = \mathfrak{B}\) is a matrix algebra, it is known [5, Theorem 2.1] that the weaker hypothesis \(\phi(R(\mathfrak{A})) \subseteq R(\mathfrak{B})\) implies that \(\phi\) is a Jordan homomorphism (i.e. preserves squares) followed by multiplication by a fixed regular element. We next show that this result does not generalize to arbitrary \(C^*\)-algebras except in a very special case, namely for commutative \(\mathfrak{B}\).

**Example.** Let \(\mathfrak{A}\) be any \(C^*\)-algebra and let \(\mathfrak{B} = M_2(\mathfrak{A})\) be the \(C^*\)-algebra of all 2 by 2 matrices with entries in \(\mathfrak{A}\). Let \(\xi\) be any automorphism of \(\mathfrak{A}\). Define a mapping \(\phi\) of \(\mathfrak{A}\) into \(M_2(\mathfrak{A})\) by the formula

\[
\phi(A) = \begin{pmatrix} A & \xi(A) \\ 0 & A \end{pmatrix}, \quad (A \in \mathfrak{A}).
\]

Then clearly \(\phi\) is a linear mapping such that \(\phi(I) = I\), but it is easy to check that \(\phi(R(\mathfrak{A})) \subseteq R(M_2(\mathfrak{A}))\) and that \(\phi\) is not a Jordan homomorphism unless \(\xi\) is the identity automorphism.

**Proposition.** Let \(\phi\) be a linear mapping of a \(C^*\)-algebra \(\mathfrak{A}\) into a commutative \(C^*\)-algebra \(\mathfrak{B}\) such that \(\phi(R(\mathfrak{A})) \subseteq R(\mathfrak{B})\). Then there is a
A \textbf{C*-homomorphism} $\rho$ of $\mathcal{A}$ into $\mathcal{B}$ and an element $B$ in $R(\mathcal{B})$ such that $\phi(A) = B\rho(A)$ for each $A$ in $\mathcal{A}$.

**Proof.** Set $\rho(A) = \phi(I)^{-1}\phi(A)$. Then $\rho(I) = I$ and $\rho(R(\mathcal{A})) \subseteq R(\mathcal{B})$ and it suffices to show that $\rho$ is a C*-homomorphism. Since $\rho(I) = I$, the condition $\rho(R(\mathcal{A})) \subseteq R(\mathcal{B})$ is equivalent to $Sp(\rho(A)) \subseteq Sp(A)$ for each $A$ in $\mathcal{A}$, where $Sp(A)$ denotes the spectrum of the operator $A$. If $U$ is any unitary operator in $\mathcal{A}$ then $Sp(\rho(U))$ is a subset of the unit circle. Since $\mathcal{B}$ is commutative, $\rho(U)$ is normal, hence unitary. The result follows from [6, Corollary 2].

We now return to the semigroup of regular contractions. By the remarks following Theorem 1 we may assume our mappings are self-adjoint.

**Lemma 3.** Let $\phi$ be a linear self-adjoint mapping of a C*-algebra $\mathcal{A}$ into a C*-algebra $\mathcal{B}$ such that $\phi(R(\mathcal{A})) \subseteq R(\mathcal{B})$ and $\phi(I) = I$. Then (i) if $P$ is a projection in $\mathcal{A}$, then $\phi(P)$ is a projection in $\mathcal{B}$; (ii) if $P$ and $Q$ are orthogonal projections in $\mathcal{A}$, then $\phi(P)$ and $\phi(Q)$ are orthogonal projections in $\mathcal{B}$.

**Proof.** (i) If $P$ is a projection, then $\phi(P)$ is a self-adjoint operator with spectrum contained in the two point set $\{0, 1\}$. (ii) If $U$ is a self-adjoint unitary operator in $\mathcal{A}$ then $\phi(U)$ is self-adjoint and unitary in $\mathcal{B}$. An operator $T$ is a projection if and only if $I - 2T$ is self-adjoint and unitary. Let $P$ and $Q$ be orthogonal projections in $\mathcal{A}$ and set $U = I - 2P$, $V = I - 2Q$. The orthogonality of $P$ and $Q$ implies that $U$ and $V$ commute. Hence $UV$ is also a self-adjoint unitary operator. Thus $\phi(UV) = I - 2(\phi(P) + \phi(Q))$ is self-adjoint and unitary so that $\phi(P) + \phi(Q)$ is a projection. It follows that $\phi(P)\phi(Q) = 0$.

**Lemma 4.** Let $\phi$ be a linear self-adjoint mapping of a commutative C*-algebra $\mathcal{A}$ into a C*-algebra $\mathcal{B}$ such that $\phi(R(\mathcal{A})) \subseteq R(\mathcal{B})$ and $\phi(I) = I$. Then $\|\phi\| = 1$.

**Proof.** Let $A$ be a positive element of $\mathcal{A}$. Then $\phi(A)$ is self-adjoint and since $Sp(\phi(A)) \subseteq Sp(A)$ it follows that $\phi(A)$ is positive. Thus $\phi$ is a positive mapping. By results of Stinespring [7, Theorems 1 and 4], there is a Hilbert space $K$, a *-representation $\rho$ of $\mathcal{A}$ on $K$ and an isometry $V$ of $H$ into $K$ ($\mathcal{B}$ acts on $H$) such that $\phi(A) = V^*\rho(A)V$ for all $A$ in $\mathcal{A}$. Thus if $A \in \mathcal{A}$, then $\|\phi(A)\| = \|V^*\rho(A)V\| \leq \|A\|$.

Recall that a von Neumann algebra is a C*-algebra which is closed in the weak operator topology [1, p. 33].

**Theorem 2.** Let $\phi$ be a linear mapping of a von Neumann algebra $M$
into a $C^*$-algebra $\mathfrak{B}$ such that $\phi(R_1(M)) \subseteq R_1(\mathfrak{B})$ and $\phi(I) = I$. Then $\phi$ is a $C^*$-homomorphism.

**Proof.** As noted above, $\phi$ is self-adjoint and $\phi(R(M)) \subseteq R(\mathfrak{B})$. Let $A$ be a self-adjoint element of $M$ of norm 1. The von Neumann algebra $M_0$ generated by $A$ is commutative and if $\epsilon > 0$ there exist orthogonal projections $P_1, P_2, \ldots, P_n$ in $M_0$ and real numbers $r_1, r_2, \ldots, r_n$ such that $\|A - \sum_1^n r_i P_i\| < \epsilon$ [1, p. 3]. By several applications of the preceding two lemmas and after a computation one obtains $\|\phi(A)^2 - \phi(A^2)\| < 2\epsilon(2 + \epsilon)$. Since $\epsilon$ was arbitrary $\phi(A)^2 = \phi(A^2)$ for each self-adjoint $A$ in $M$ of norm 1. It follows trivially that $\phi$ is a $C^*$-homomorphism.

We note that the theorem holds with an identical proof in case $M$ is an $AW^*$-algebra [4].

**Remarks.** 1. It is an open question as to whether Theorem 2 is true when $M$ is a $C^*$-algebra. Since the conclusion, i.e. $\phi(A)^2 = \phi(A^2)$, need only hold for self-adjoint operators $A$, there is no loss of generality in assuming $M$ to be commutative. Then by Lemma 4 we have $\|\phi\| = 1$.

2. The author believes that a solution to the following special case would shed considerable light on the problem: let $\mathfrak{A}$ be a commutative $C^*$-algebra acting on a Hilbert space $H$, and let $P$ be a projection operator on $H$, say mapping $H$ onto a subspace $K$. Let $\phi$ be the mapping $\phi(A) = PA$ of $\mathfrak{A}$ into the bounded operators on $K$. The reason for this belief is the relation of the mapping $A \rightarrow PA$ to the results of Stinespring quoted above, and to normal dilations of operators.

**References**


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