J. Witkowski in [3] proved a theorem of S. Golab which gives a characterization of the sphere in $E^3$. In this paper a simpler proof of Golab’s theorem is presented. The more direct approach involved should make the geometry simpler to visualize. First we state necessary

Definitions. (I) A curve on a surface of class $C^1$ is called $B$-straight if the tangent planes to the surface along the curve remain always perpendicular to a fixed direction. (II) A curve on a surface of class $C^1$ is called $B$-plane if the tangent planes to the surface along the curve are all parallel to a fixed direction.

We wish to prove the following

**Theorem.** If every geodesic of a regular surface of class $C^3$ is $B$-plane but not $B$-straight, then the surface is part of a sphere.

**Proof (indirect).** According to the hypothesis of our theorem each geodesic is $B$-plane, that is there exists for each geodesic a constant unit vector $V$ which is perpendicular to the surface normal $N$ along the geodesic. Differentiation of $V \cdot N = 0$ with respect to the arc length yields $V \cdot d_sN = 0$ where $d_sN$ is not identically zero since by hypothesis the geodesic is not $B$-straight. We assume that $d_sN \neq 0$ at the points under discussion and the later development will show that other points need not be considered. Now for a geodesic, the principal normal $n$ is equal to the surface normal $N$. Thus the unit tangent $t$ satisfies the relation $V \cdot t = \cos \theta = \text{const}$ which can be shown by differentiation. This means that the geodesics are helices, some fixed angle $\theta$ belonging to each geodesic. Also, at a point of a geodesic the vectors $V$, $t$, and $d_sN$ are in the tangent plane and we find for the curvature $\kappa$, which along a geodesic is the same as the normal curvature,

\begin{equation}
-\kappa = t \cdot d_sN = \pm |d_sN| \sin \theta.
\end{equation}

Using the third fundamental form [2, p. 103] along a geodesic, we have

\begin{equation}
d_sN \cdot d_sN = -\kappa_1\kappa_2 + (\kappa_1 + \kappa_2)\kappa,
\end{equation}

which allows us to change (1) to

Received by the editors November 16, 1965.
\[ \kappa^2 = \sin^2 \theta [ -\kappa_1 \kappa_2 + (\kappa_1 + \kappa_2) \kappa ] . \]

Assume the existence of a neighborhood \( R_1 \) on the surface that does not contain any umbilic points and therefore is covered by lines of curvature. Within \( R_1 \) there must be a neighborhood \( R_2 \) where the curvature \( K \) does not vanish. If such an \( R_2 \) would not exist, the set of points in \( R_1 \) with \( K = 0 \) would be dense and a continuity argument would show that \( K = 0 \) at all points of \( R_1 \). In this case, however, we have geodesics that are \( B \)-straight [3, Lemma 1]. Such geodesics being ruled out by our hypothesis we now take a point \( P \) in \( R_2 \). Consider the geodesic through \( P \) in the principal direction corresponding to \( \kappa_1 \). Its curvature at \( P \) is also \( \kappa_1 \) which we know to be different from zero. But then (2) shows that \( d_s N \neq 0 \) and consequently the arguments leading to (3) are valid. We can infer from (3) that \( \sin^2 \theta = 1 \). Then along this geodesic \( \sin^2 \theta \) will continue to equal 1 and continuity shows that the value of \( \kappa \) found in (3) will be equal to \( \kappa_1 \) throughout. Hence the geodesic coincides with the line of curvature in \( R_2 \). Also, in \( R_2 \) the lines of curvature can be used as coordinate curves [1, p. 56]. Since they are geodesics, \( R_2 \) has curvature \( K = 0 \) [1, p. 45].

It follows that \( K \) is identically zero in \( R_1 \). Therefore, a neighborhood \( R_1 \) without umbilics satisfying the hypothesis of our theorem cannot exist. Rather, the set of umbilics on the surface is dense and we are dealing with part of a sphere [3, Lemma 2].

**References**


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