COEFFICIENT ESTIMATES FOR STARLIKE FUNCTIONS OF ORDER \( \alpha \)

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In [1] MacGregor obtained upper bounds for the moduli of the coefficients of a function \( z + \sum_{n=0}^{\infty} a_n z^n \) which is starlike in the unit disc. The purpose of this note is to extend MacGregor's result to the class of starlike functions of order \( \alpha \) introduced by Robertson, [2], and to obtain an improvement on this when \( f(z) \) is bounded in the unit disc.

**Definition.** A function \( f(z) \) is said to be starlike of order \( \alpha \), \((0 \leq \alpha < 1)\), if it is univalent and \( \Re \{zf'(z)/f(z)\} \geq \alpha \) for \( |z| < 1 \).

**Theorem.** If \( f(z) = z + \sum_{n=0}^{\infty} a_n z^n \) is starlike of order \( \alpha \) then

\[
|a_n| \leq \left( \frac{(2-2\alpha)/k + m - 1}{m} \right)
\]

where \( mk+1 \leq n \leq mk+k \), \( m = 1, 2, 3, \ldots \). If, further, \( |f(z)| < 1 \) for \( |z| < 1 \) then we also have

\[
\sum_{n=p-k+1}^{p} (n + 1 - 2\alpha)^2 |a_n|^2 \leq 4(1 - \alpha)^2 \quad \text{for} \quad p \geq k.
\]

**Lemma 1.**

\[
4(1 - \alpha) \left\{ 1 - \alpha + \sum_{m=1}^{q-1} (mk+1-\alpha) \left[ \frac{1}{m!} \prod_{\mu=0}^{m-1} \left( \mu + \frac{2-2\alpha}{k} \right) \right] \right\}
\]

\[
= \left\{ \frac{k}{(q-1)!} \prod_{\mu=0}^{q-1} \left( \mu + \frac{2-2\alpha}{k} \right) \right\}^2 \quad \text{for} \quad q = 2, 3, \ldots
\]

This is easily proved by induction on \( q \).

**Lemma 2.** If \( k = 1, 2, \ldots, q = 1, 2, \ldots \), and \( \alpha < 1 \) then

\[
(n - 1)^2 \geq (qk)^2 (n - \alpha)/(qk + 1 - \alpha) \quad \text{for} \quad n \geq qk + 1.
\]

**Proof of Theorem.** By the method of [1], if \( g(z) = zf'(z)/f(z) \) and \( h(z) = (g(z) - 1)/(g(z) + 1 - 2\alpha) = b_k z^k + b_{k+1} z^{k+1} + \cdots \) then \( (n-1)a_n = 2(1-\alpha)b_n \) for \( n = k+1 \) to \( 2k \), and

\[
\sum_{n=k+1}^{2k} (n - 1) |a_n|^2 \leq 4(1 - \alpha)^2.
\]

Received by the editors March 16, 1966.

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Also, for some set of constants \( d_n \),
\[
\sum_{n=k+1}^{p} (n - 1) d_n z^n + \sum_{n=p+1}^{\infty} d_n z^n = h(z) \left\{ 2(1 - \alpha) z + \sum_{n=k+1}^{p-k} (n + 1 - 2\alpha) d_n z^n \right\}.
\]
Since \(|h(z)| < 1\) for \(|z| < 1\) it then follows as in the original paper [3] of Clunie that, for \( 0 \leq r < 1 \),
\[
\sum_{n=k+1}^{p} (n - 1)^2 \left| a_n \right|^2 r^{2n} \leq 4(1 - \alpha)^2 r^2 + \sum_{n=k+1}^{p-k} (n + 1 - 2\alpha)^2 \left| a_n \right|^2 r^{2n}.
\]
Since \( a_1 = 1 \) and \( a_2 = \cdots = a_k = 0 \) this may be written as
\[
\sum_{n=p-k+1}^{p} (n - 1)^2 \left| a_n \right|^2 r^{2n} \leq 4(1 - \alpha) \sum_{n=1}^{p-k} (n - \alpha) \left| a_n \right|^2 r^{2n}.
\]
Letting \( r \) tend to 1 yields
\[
\sum_{n=p-k+1}^{p} (n - 1)^2 \left| a_n \right|^2 \leq 4(1 - \alpha) \left\{ 1 - \alpha + \sum_{n=k+1}^{p-k} (n - \alpha) \left| a_n \right|^2 \right\}.
\]
We next consider
\[
\sum_{n=mk+1}^{mk+k} (n - 1) \left| a_n \right|^2 \leq \left\{ \frac{k}{(m - 1)!} \prod_{\mu=0}^{m-1} \left( \mu + \frac{2 - 2\alpha}{k} \right) \right\}^2
\]
and
\[
\sum_{n=mk+1}^{mk+k} (n - \alpha) \left| a_n \right|^2 \leq (mk + 1 - \alpha) \left\{ \frac{1}{m!} \prod_{\mu=0}^{m-1} \left( \mu + \frac{2 - 2\alpha}{k} \right) \right\}^2.
\]
In the case \( m = 1 \), (A) reduces to equation (i) and (B) follows by an application of Lemma 2 to (A). For \( m = q > 1 \), (A) and (B) are proved inductively, as in [1], by applying (B) with \( m = 1 \) to \( q - 1 \) and Lemma 1 to (iii) with \( p = (q+1)k \), and Lemma 2 to (A) with \( m = q \).

From (A) it follows that, for \( mk+1 \leq n \leq mk+k \),
\[
\left| a_n \right| \leq \frac{k}{(n - 1)(m - 1)!} \prod_{\mu=0}^{m-1} \left( \mu + \frac{2 - 2\alpha}{k} \right)
\leq \frac{1}{m!} \prod_{\mu=0}^{m-1} \left( \mu + \frac{2 - 2\alpha}{k} \right).
\]
It should be noted that this result

$$|a_n| \leq \left( \frac{(2 - 2\alpha)/k + m - 1}{m} \right)$$

is "sharp" for $n = mk + 1$, $(m = 1, 2, \ldots)$, for the function $f(z) = z(1 - z^k)^{-2(1 - \alpha)/k}$; and that when $\alpha = 0$ it gives the same bounds for the coefficients as Waadeland [4] found in the case of $k$-symmetric univalent functions.

We also have, from (ii), that

$$\sum_{n=p-k+1}^{p} (n + 1 - 2\alpha)^2 |a_n|^2r^{2n} \leq 4(1 - \alpha) \sum_{n=1}^{p} (n - \alpha) |a_n|^2r^{2n}$$

$$\leq 4(1 - \alpha) \sum_{n=1}^{\infty} (n - \alpha) |a_n|^2r^{2n}.$$ 

As on page 232 of [5], the right hand-side does not exceed

$$\frac{2(1 - \alpha)}{\pi} \int_{0}^{2\pi} \left\{ |\text{Re} \frac{zf'(z)}{f(z)} - \alpha| \right\} d\theta$$

where $z = re^{i\theta}$

$$= 4(1 - \alpha)^2.$$

It now follows that

$$\sum_{n=p-k+1}^{p} (n + 1 - 2\alpha)^2 |a_n|^2 \leq 4(1 - \alpha)^2 \quad \text{for} \quad p \geq k.$$

The author would like to thank the referee for his useful comments and his reference to [4].

References


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