ON EXTENSIONS OF CAYLEY ALGEBRAS

MARGARET M. HUMM AND ERWIN KLEINFELD

Kaplansky in Theorem 2 of [3] has shown that if \( A \) is an alternative algebra with identity element 1 which contains a subalgebra \( B \) isomorphic to a Cayley algebra and if 1 is contained in \( B \) then \( A \) is isomorphic to the Kronecker product \( B \otimes T \), where \( T \) is the center of \( A \). Jacobson in Theorem 2 of [2] has shown that if \( A \) is an alternative algebra which contains a subalgebra \( B \) isomorphic to a Cayley algebra, then the identity \( e \) of \( B \) must lie in the center of \( A \), provided \( A \) has characteristic different from 2. He also has given a new proof of the Kaplansky result, using his classification of completely reducible alternative bimodules. In the present note we present a generalization of the aforesaid result by Jacobson, which incidentally is also valid for characteristic 2.

**Theorem.** Let \( A \) be an alternative algebra over \( F \) and \( B \) any subalgebra with identity \( e \). Then consider the following two conditions.

(i) There exist \( x, y \) in \( B \), \( \alpha \) in \( F \) such that \( e = \alpha(x, y)^4 \), where \( (x, y) = xy - yx \).

(ii) The ideal \( I \) of \( B \), generated by all associators of \( B \) equals \( B \). If \( B \) satisfies (i) then \( e \) must be in the nucleus \( N \) of \( A \). If \( B \) satisfies (i) and (ii) then \( e \) must be in the center \( C \) of \( A \).

**Proof.** It will be helpful to recall some identities that hold in all alternative rings \( R \). Let \( p, q, r, s, t, x, y, z \) be arbitrary elements of \( R \) and \( n \) an arbitrary element of the nucleus \( N' \) of \( R \). Then

1. \((s, t)^4\) is in \( N' \),
2. \((n, r)\) is in \( N' \),
3. \((n, (x, y, z)) = 0 \),
4. \((n, r)(x, y, z) = -(n, x)(r, y, z) \),
5. \((p^2, q) = p(p, q) + (p, q)p \).

A proof of (1) may be found in Theorem 3.1 (ii) of [5]. Proofs of (2), (3) and (4) are contained in Lemma 2.3 (ii), (iii) and (iv) of [4]. Identity (5) may be verified directly by expanding both sides of the equation and using the alternative law. If \( B \) satisfies the hypothesis and condition (i), then one may apply (1) directly to obtain that \( e \)
belongs to $N$. If $B$ also satisfies condition (ii), then select $n = e, r$ as arbitrary in $A$ and $x, y, z$ arbitrary in $B$ and substitute this in (4). Then $(e, r)(x, y, z) = -(e, x)(r, y, z) = 0$, since $(e, x) = 0$. The associator ideal $I$ of $B$ may be characterized as the additive subgroup of $B$ generated by all elements of the form $(B, B, B)$ and $(B, B, B)B$. We have already proved that $(e, r)(B, B, B) = 0$. But $(e, r)$ belongs to $N$ as a result of (2), so that $(e, r) \cdot (B, B, B)B = 0$ is also obvious and hence $(e, r)I = 0$. Since $I = B$ and $e$ itself belongs to $B$, we have $(e, r)e = 0$. Using (2) we may substitute $n = (e, r)$ in (3) to obtain also that $(B, B, B)(e, r) = 0$. As $I$ may also be characterized as the additive subgroup generated by elements of the form $(B, B, B)$ and $B(B, B, B)$, we obtain $I(e, r) = 0$, and hence $e(e, r) = 0$. At this point we substitute $p = e, q = r$ in (5) and obtain $(e, r) = (e^2, r) = e(e, r) + (e, r)e = 0$. This places $e$ in $C$ and the proof of the theorem is complete.

Condition (i) certainly holds when $B$ is taken to be a quaternion algebra and hence a priori if $B$ is a Cayley algebra. Since Cayley algebras are simple and not associative, condition (ii) clearly holds when $B$ is taken to be a Cayley algebra. Thus we obtain Jacobson's result as a corollary to our theorem. On the other hand one may readily construct other alternative algebras to which our theorem applies.

We conclude with an example that shows a quaternion algebra may be embedded as a subalgebra of an associative algebra and with the identity quaternion not in the center of the larger algebra. Consider the free associative algebra $S$ on the four generators $w, x, y, z$. Define relations on $x, y, z$ which make them behave as the quaternions $1, i, j$ respectively. In the quotient algebra $R$, words have the form

$$\cdots q_1 w^{k_1} \cdots q_n w^{k_n} \cdots$$

where $q_i = \pm x, \pm y, \pm z, \pm yz$. Then $R$ contains a copy of the quaternions with identity $x$, but $wx \neq xw$, so that $x$ is not in the center of $R$. If an example that is alternative but not associative is desired, then one may take a direct product of $R$ with a Cayley algebra.

Bibliography

A CONDITION FOR A FINITE GROUP TO BE NILPOTENT

STEPHEN MONTAGUE AND GOMER THOMAS

Let $\mathcal{C}$ be a class of groups such that:

(i) If $G$ is in $\mathcal{C}$, then every homomorphic image of $G$ is in $\mathcal{C}$.

(ii) If $G$ is finite and $G/\phi(G)$ is in $\mathcal{C}$, where $\phi(G)$ is the Frattini subgroup of $G$, then $G$ is in $\mathcal{C}$.

Examples of such classes are the class of nilpotent groups and the class of supersolvable groups. Others can be found in a paper by Baer [1].

In this note a theorem of P. Hall on nilpotent groups is proved as a corollary to the following:

**Theorem.** If $G$ is a finite group with a subgroup $H$ such that $\phi(H)$ is normal in $G$ and $G/\phi(H)$ is in $\mathcal{C}$, then $G$ is in $\mathcal{C}$.

**Lemma (Huppert).** Let $G$ be a finite group, $H$ be a subgroup of $G$, and $N$ be a subgroup of $H$ such that $N$ is normal in $G$ and $N \leq \phi(H)$. Then $N \leq \phi(G)$.

**Proof.** If not, $G$ would have to have a maximal subgroup $U$ such that $N \nsubseteq U$. Then $H = G \cap H = NU \cap H = N(U \cap H) = U \cap H$, since $N \leq \phi(H)$. But this implies $H \leq U$, contrary to $N \nsubseteq U$.

**Proof of Theorem.** An application of the Lemma with $N = \phi(H)$ shows that $\phi(H) \leq \phi(G)$. Hence $G/\phi(G)$ is in $\mathcal{C}$, and so $G$ is in $\mathcal{C}$.

**Corollary.** If $G$ is a finite group with a normal subgroup $H$ such that $H$ is nilpotent and $G/H'$ is nilpotent, where $H'$ is the commutator subgroup of $H$, then $G$ is nilpotent.

**Proof.** Since $H$ is nilpotent, $\phi(H)$ contains $H'$. Hence $G/\phi(H)$ is

Received by the editors September 22, 1965.