ON THE MULTIPLICATION OF TENSOR FIELDS

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Let $M$ be a paracompact $n$-dimensional manifold of class $C^{k+1}$, $T(x)$ the tangent space of the point $x \in M$ and $T(x)^* \ast$ the dual space. A real valued $(p+r)$-linear function $\Phi(x)$ with $p$ arguments in $T(x)$ and $r$ arguments in $T(x)^* \ast$ is called a tensor of type $(p, r)$ at the point $x$. The tensors of type $(p, r)$ at $x$ form a linear space $T_p^r(x)$. The products of two tensors $\Phi \in T_p^r(x)$ and $\Psi \in T_q^s(x)$ is defined by

$$(\Phi \Psi)(x; \xi_1, \ldots, \xi_{p+q}, \xi^*_{1}, \ldots, \xi^*{p+r}) = \Phi(x; \xi_1, \ldots, \xi_p, \xi^*_{1}, \ldots, \xi^*{r}) \cdot \Psi(x; \xi_{p+1}, \ldots, \xi_{p+q}, \xi^*{r+1}, \ldots, \xi^*{r+s}),$$

$\xi_x \in T(x), \xi^*_u \in T(x)^* \ast$.

A tensor field of type $(p, r)$ and class $C^k$ on $M$ is an assignment of tensors of type $(p, r)$ to the points of $M$ such that the components with respect to a local coordinate system are $C^k$-functions. The set of all tensor fields of type $(p, r)$ and class $C^k$ is a module $T_p^r$ over the ring $F$ of $C^k$-functions on $M$. The multiplication of tensors induces a multiplication of tensor fields in an obvious way. Now consider the $F$-bilinear mapping

$$T_p^r \times T_q^s \to T_{p+q}^{r+s}$$

which is defined by the multiplication. This bilinear mapping induces a $F$-linear mapping

$$h: T_p^r \otimes T_q^s \to T_{p+q}^{r+s}$$

such that

$$h(\Phi \otimes \Psi) = \Phi \cdot \Psi.$$  

We shall prove in this paper the following

**Theorem.** $h: T_p^r \otimes T_q^s \to T_{p+q}^{r+s}$ is an isomorphism.

For the sake of simplicity we restrict ourselves to covariant tensor...
fields, i.e. tensor fields of type \((p, 0)\) and write \(T_p\) instead of \(T^p_0\). However, the argument can be carried over word by word to the general case.

Before giving the proof of the theorem we state some corollaries and show how they can be deduced from the theorem. We are indebted to the referee for the suggestion to include the Corollaries 3 and 4.

**Corollary 1.** Let \((T^r_p)^*\) be the dual of the \(F\)-module \(T^r_p\). Then

\[(T^r_p)^* \otimes (T^s_q)^* \cong (T^r_p \otimes T^s_q)^* .\]

**Proof.** There exists a canonical \(F\)-isomorphism

\[\phi_p: (T^r_p)^* \to T^r_p.\]


\[\begin{array}{rcl}
(T^r_p)^* \otimes (T^s_q)^* & \xrightarrow{\phi_p \otimes \phi_q^*} & T^r_p \otimes T^s_q \\
& \xrightarrow{h} & T^{r+s}_{r+s} \\
& \downarrow i & \\
(T^r_p \otimes T^s_q)^* & \xrightarrow{h^*} & (T^{r+s}_{r+s})^* \end{array}\]

are isomorphisms it follows that

\[(h^*)^{-1} \circ (\phi_{p+q}^*)^{-1} \circ h \circ (\phi_p^* \otimes \phi_q^*)\]

is an isomorphism of \((T^r_p)^* \otimes (T^s_q)^*\) onto \((T^r_p \otimes T^s_q)^*\).

**Corollary 2.** The abstract \(p\)th tensorial power \(\otimes^p T_1\) is isomorphic to \(T_p\) under \(h\).

**Proof.** This follows immediately from the theorem.

**Symmetric Tensors.** Let \(S: \otimes^p T_1 \to \otimes^p T_1\) be the operator of symmetry defined by

\[S(\omega^1 \otimes \cdots \otimes \omega^p) = \frac{1}{p!} \sum_\sigma \omega^{(1)} \otimes \cdots \otimes \omega^{(p)}\]

where \(\sigma\) runs through all permutations of \(p\) objects. The symmetric product \(V^p T_1\) is defined to be the \(F\)-module \(\text{Im} S \subset \otimes^p T_1\). Denote by \(i: V^p T_1 \to \otimes^p T_1\) the inclusion homomorphism. On the other hand consider the submodule \(S_p \subset T_p\) of symmetric tensors; let \(i': S_p \to T_p\).
be the inclusion homomorphism. \( \Phi \) is in \( S_p \) if and only if for any permutation \( \sigma \) and any vector fields \( \xi_1, \ldots, \xi_p \) we have \( (\sigma \Phi)(\xi_1, \ldots, \xi_p) = \Phi(\xi_{\sigma(1)}, \ldots, \xi_{\sigma(p)}) = \Phi(\xi_1, \ldots, \xi_p) \).

**Corollary 3.** The \( F \)-isomorphism \( h: \bigotimes^p T_1 \rightarrow T_p \) induces an \( F \)-isomorphism \( \tilde{h}: V^p T_1 \rightarrow S_p \) such that \( h \circ i = i' \circ \tilde{h} \).

**Proof.** One easily verifies that \( \text{Im}(h \circ i) = \text{Im} i' \); then defines \( \tilde{h} = i'^{-1} \circ h \circ i \). q.e.d.

**Exterior Forms.** The \( p \)th exterior power \( \Lambda^p T_1 \) is the quotient \( F \)-module \( \bigotimes^p T_1 / N \) where \( N \subseteq \bigotimes^p T_1 \) is the submodule generated by the elements \( \Phi \in \bigotimes^p T_1 \) such that \( \tau \Phi = \Phi \) for some transposition \( \tau \). The operator \( \alpha \) of antisymmetrization is defined by

\[
\alpha \Phi = \frac{1}{p!} \sum_{\sigma} \epsilon_\sigma \cdot \sigma \Phi,
\]

where \( \sigma \) runs over all permutations of \( p \) objects and \( \epsilon_\sigma \) is the sign of the permutation \( \sigma \). Let \( N_1 \subseteq \bigotimes^p T_1 \) be the kernel of \( \alpha \). According to Bourbaki, *Algèbre*, Chapter III, p. 60, 2e edition, we have \( N \subseteq N_1 \). But on the other hand by Proposition 3 in Bourbaki, loc. cit., p. 58, for any \( \Phi \in \bigotimes^p T_1 \) and any permutation \( \sigma \), we have \( \Phi = \epsilon_\sigma \cdot \sigma \Phi \in N \).

Therefore

\[
\sum_{\sigma} (\Phi - \epsilon_\sigma \cdot \sigma \Phi) \in N
\]

or \( p! \Phi - \alpha \Phi \in N \). Hence, if \( \Phi \in N_1 \), then \( \Phi \in N \). Whence \( N_1 = N \).

Now let \( h: \bigotimes^p T_1 \rightarrow T_p \) be the isomorphism of our main theorem, \( \pi: \bigotimes^p T_1 \rightarrow \Lambda^p T_1 \) the canonical projection homomorphism and \( \alpha': T_p \rightarrow A_p \subseteq T_p \) the antisymmetrization map in \( T_p \). \( A_p \) consists of the antisymmetric tensors or global \( p \)-forms.

**Corollary 4.** \( h: \bigotimes^p T_1 \rightarrow T_p \) induces an \( F \)-isomorphism \( \tilde{h}: \Lambda^p T_1 \rightarrow A_p \) such that \( \tilde{h} \circ \pi = \alpha' \circ h \).

**Proof.** We have shown that \( N_1 = N \) or equivalently \( \ker \alpha = \ker \pi = N \). It is easy to verify by computation that \( h \circ \alpha = \alpha' \circ h \). Then \( \ker \pi = \ker \alpha = \ker \alpha' \) and this proves that \( \tilde{h} \) exists and is an isomorphism.

We proceed to the proof of the main theorem.

**Lemma I.** Let \( U_\alpha \) be a system of coordinate neighborhoods on \( M \) such that

\[
U_\alpha \cap U_\beta = \emptyset \quad \text{if} \quad \alpha \neq \beta.
\]
and \( A_\alpha \) be a compact subset of \( U_\alpha \). Then there exists a system of \( n \) tensor fields \( \omega^i \in T_1 \) such that the \( n \) tensors \( \omega^i(x) \) are linearly independent for every \( x \in U_\alpha \cap A_\alpha \).

**Proof.** In each \( U_\alpha \) there exists a system of \( n \) tensor fields \( \tilde{\omega}_\alpha^i \) of order 1 in \( U_\alpha \) such that the tensors \( \tilde{\omega}_\alpha^i(x) \), \((i = 1 \cdots n) \) are linearly independent at every point \( x \in U_\alpha \). Now let \( h_\alpha \) be a \( C^k \)-function on \( M \) such that the carrier of \( h_\alpha \) is compact and contained in \( U_\alpha \) and that

\[
h_\alpha = 1 \quad \text{in} \quad A_\alpha.
\]

Define \( \omega_\alpha^i \) by

\[
\omega_\alpha^i = \begin{cases} h_\alpha \tilde{\omega}_\alpha^i & \text{in} \quad U_\alpha, \\ 0 & \text{in} \quad M - U_\alpha \end{cases}
\]

and \( \omega^i \) by

\[
\omega^i = \sum_\beta \omega^i_\beta.
\]

Then \( \omega^i \) is a tensor field of order 1 on \( M \). Now let \( x \in U_\alpha \cap A_\alpha \) be an arbitrary point. Since the compact sets \( A_\alpha \) are mutually disjoint the point \( x \) belongs to precisely one of them, say to \( A_\alpha \). This implies that

\[
\omega^i(x) = h_\alpha(x)\tilde{\omega}_\alpha^i(x) = \tilde{\omega}_\alpha^i(x)
\]

i.e., the tensors \( \omega^i(x) \) are linearly independent.

**Lemma II.** Consider the sets \( U_\alpha \) and \( A_\alpha \) given in Lemma I. Then there exists a system of \( n \) tensor fields \( \omega^i \in T_1 \) with the following property: Every tensor field \( \phi \in T_p \) whose carrier is contained in \( U_\alpha \cap A_\alpha \) can be written in the form

\[
\phi = \sum_{(\rho)} \lambda_{\rho_1 \cdots \rho_p} \omega^{\rho_1} \cdots \omega^{\rho_p}
\]

where the \( \lambda_{\rho_1 \cdots \rho_p} \) are scalar functions on \( M \) whose carriers are contained in \( U_\alpha \cap A_\alpha \).

**Proof.** Choose a system of open sets \( B_\alpha \) with compact closure such that

\[
A_\alpha \subset B_\alpha \subset \overline{B}_\alpha \subset U_\alpha.
\]

Applying Lemma I to the compact sets \( \overline{B}_\alpha \) we obtain \( n \) tensor fields \( \omega^i \in T_1 \) such that the tensors \( \omega^i(x) \) are linearly independent for every
$x \in \bigcup_a B_a$. Hence, we can write

\[(1) \quad \phi(x) = \sum_{(r)} \lambda_{r_1 \ldots r_p}(x) \omega^{r_1}(x) \cdots \omega^{r_p}(x), \quad x \in \bigcup_a B_a,\]

where the coefficients are $C^k$-functions in $\bigcup_a B_a$. Since the carrier of $\Phi$ is contained in $\bigcup_a A_a$ the same must be true for every function $\lambda_{r_1 \ldots r_p}$. Hence, a system of $C^k$-functions $\lambda_{r_1 \ldots r_p}$ can be defined on $M$ by

\[\lambda_{r_1 \ldots r_p} = \begin{cases} \lambda_{r_1 \ldots r_p} & \text{in } \bigcup_a B_a, \\ 0 & \text{in } M - \bigcup_a B_a. \end{cases}\]

Then

\[(2) \quad \Phi(x) = \sum_{(r)} \lambda_{r_1 \ldots r_p}(x) \omega^{r_1}(x) \cdots \omega^{r_p}(x)\]

for every point $x \in M$. In fact, if $x \in \bigcup_a B_a$, the relation (2) follows from (1) and otherwise both sides of (2) are zero.

**Lemma III.** With $U_a$ and $A_a$ as in Lemma I consider any 2r tensor fields $\Phi^i \in T_p$ and $\Psi^j \in T_p$ where the carriers of the $\Phi^i$ are contained in $\bigcup_a A_a$. Then the relation

\[(3) \quad \sum_i \Phi^i \cdot \Psi^j = 0\]

implies that

\[\sum_i \Phi^i \otimes \Psi^j = 0.\]

**Proof.** Choose the $B_a$ as in Lemma II and let $\omega^i$ be the tensor fields constructed in Lemma II. Then $\Phi^i$ can be written as

\[\Phi^i = \sum_{(r)} \lambda^i_{r_1 \ldots r_p} \omega^{r_1} \cdots \omega^{r_p}.\]

It follows from (3) that

\[(4) \quad \sum_i \sum_{(r)} \lambda^i_{r_1 \ldots r_p} \omega^{r_1} \cdots \omega^{r_p} \cdot \Psi^j = 0.\]

Since the tensors $\omega^i(x)$ are linearly independent for every $x \in \bigcup_a B_a$ the relation (4) implies that
by Lemma II the carrier of $\lambda_{r_1, \ldots, r_p}$ is contained in $\bigcup_a A_a$ and hence in $\bigcup_a B_a$. Thus (5) holds for every $x \in M$, i.e.

$$\sum_j \lambda_{r_1, \ldots, r_p}^j \psi^j = 0.$$ 

Now the bilinearity of the tensor product yields

$$\sum_i \Phi^i \otimes \psi^j = \sum_i \left( \sum_{(r)}^{j} \lambda_{r_1, \ldots, r_p} \omega^{r_1} \cdots \omega^{r_p} \right) \otimes \psi^j$$

$$= \sum_{(r)} \left( \omega^{r_1} \cdots \omega^{r_p} \otimes \sum_j \lambda_{r_1, \ldots, r_p}^j \psi^j \right) = 0.$$

**Lemma IV.** Let $M$ be a paracompact $n$-dimensional manifold. Then there exists a locally finite covering by open sets $V^k_\alpha$ where $k = 0, 1, \ldots, n_0$ ($n_0 \leq n$) and $\alpha \in \mathcal{I}_k$ ($\mathcal{I}_k$ index sets) subject to the following conditions:

(i) $V^k_\alpha$ is compact,

(ii) $V^k_\alpha$ is contained in a coordinate neighborhood,

(iii) $V^k_\alpha \cap V^k_\beta = \emptyset$ for $\alpha \neq \beta$.

**Proof.** Since $M$ is a manifold, we may consider the covering $\{U\}$ consisting of all relatively compact coordinate neighborhoods. $M$ is paracompact and hence there is a locally-finite refinement $\{S\}$ of $\{U\}$. As a paracompact space, $M$ is normal; $M$ has dimension $n$, hence $\{S\}$ has a refinement $\{R_\mu\}$ of order $\leq n$. (See C. H. Dowker, Amer. J. Math. (1947), p. 211, together with W. Hurewicz, *Dimension theory*, Princeton Univ. Press, Princeton, N. J., 1941; Theorem V8, p. 67.) Again, since $M$ is paracompact, there is a locally finite refinement $\{W_\mu\}$ of $\{R_\mu\}$ with index set a subset of the former index set and $W_\mu \subset R_\mu$. For if $\{Z_\beta\}$ is a locally-finite refinement of $\{R_\mu\}$ choose $\mu(\beta)$ such that $Z_\beta \subset R_{\mu(\beta)}$ and put $W_\mu = \bigcup_{\mu(\beta) = \mu} Z_\beta$. Then $\{W_\mu\}$ is locally finite and of order $n_0 \leq n$. There exists a partition of unity $\{\phi_\mu\}$ with carrier $\phi_\mu \subset W_\mu$. Of course carrier $\phi_\mu$ is compact. Given $k+1$ different indices $\mu_0, \ldots, \mu_k$ put $\alpha = (\mu_0 \cdots \mu_k)$ and consider the sets

$$V^k_\alpha = \{x \mid x \in M, \phi_\mu(x) < \min [\phi_{\mu_0}(x), \cdots, \phi_{\mu_k}(x)] \mu \neq \mu_0, \ldots, \mu_k\}.$$ 

Each $V^k_\alpha$ is open and $V^k_\alpha \cap V^k_\beta = \emptyset$ for $\alpha \neq \beta$. Furthermore,

$$V^k_\alpha \subset \text{(carrier } \phi_{\mu_0}) \cap \cdots \cap \text{(carrier } \phi_{\mu_k}).$$
Hence $V^k_\alpha$ is compact and contained in some $W_\mu$. Therefore it is contained in a coordinate neighborhood. Since the order of the covering \{ $W_\mu$ \} is $n_0$, for $k > n_0$ the sets $V^k_\alpha$ are void. The sets $V^k_\alpha$ ($0 \leq k \leq n_0$) cover $M$ since for every $x \in M$ some $\phi_\mu(x) > 0$. The covering \{ $V^k_\alpha$ \} is locally finite since \{ $W_\mu$ \} is and hence it has all desired properties.

**Theorem.** The homomorphism $h$ is an isomorphism onto $T_{p+q}$.

**Proof.** Consider the covering

$$ M = \bigcup_{\alpha} \bigcup_{k=0}^{n_0} V^k_\alpha $$

constructed in Lemma IV. Since $M$ is paracompact and the covering \{ $V^k_\alpha$ \} is locally finite we can choose an open subset $W^k_\alpha$ in each $V^k_\alpha$ such that

$$ W^k_\alpha \subseteq V^k_\alpha $$

and

$$ \bigcup_{\alpha} \bigcup_{k=0}^{n_0} W^k_\alpha = M. $$

It follows from (7) and the property (i) in Lemma IV that the closures $\overline{W}^k_\alpha$ are compact.

Put

$$ W^k = \bigcup_{\alpha} W^k_\alpha \quad (k = 0 \cdots n) $$

and let $f^k$ be a partition of unity subordinate to the covering \{ $W^k$ \}. Given an arbitrary tensor field $\Omega \in T_{p+q}$ consider the tensor fields $\Omega^k = f^k \Omega$. The carrier of $\Omega^k$ is contained in $W^k$. Applying Lemma II with

$$ U_\alpha = V^k_\alpha \quad \text{and} \quad A_\alpha = \overline{W}^k_\alpha, $$

we see that $\Omega^k$ can be written as

$$ \Omega^k = \sum_{(r)} \lambda^k_{r_1 \cdots r_{p+q}} \omega^{r_1} \cdots \omega^{r_{p+q}} $$

where $\omega^r \in T_1$ and $\lambda^k_{r_1 \cdots r_{p+q}} \in F$. Introducing the tensor fields

$$ \Phi^{r_1 \cdots r_p} = \omega^{r_1} \cdots \omega^{r_p}, $$
we obtain from (9)

\[
\Omega^k = \sum_{(\nu)} \lambda_{\nu_1 \cdots \nu_{p+q}}^k \Phi^{\nu_1 \cdots \nu_p} \Psi^{\nu_{p+1} \cdots \nu_{p+q}}
\]

\[
= h \left( \sum_{(\nu)} \lambda_{\nu_1 \cdots \nu_{p+q}}^k \Phi^{\nu_1 \cdots \nu_p} \otimes \Psi^{\nu_{p+1} \cdots \nu_{p+q}} \right).
\]

Summation over \( K \) yields

\[
\Omega = h \left( \sum_{k=0}^{n} \sum_{(\nu)} \lambda_{\nu_1 \cdots \nu_{p+q}}^k \Phi^{\nu_1 \cdots \nu_p} \otimes \Psi^{\nu_{p+1} \cdots \nu_{p+q}} \right).
\]

This relation shows that \( h \) is an onto map.

To prove that \( h \) is one-to-one suppose that

\[
h \left( \sum \Phi^j \otimes \Psi^j \right) = 0
\]

where

\[
\Phi^j \in T_p \quad \text{and} \quad \Psi^j \in T_q.
\]

Then \( \sum \Phi^j \otimes \Psi^j = 0 \) and multiplication by \( f^k \) yields

\[
\sum_j f^k \Phi^j \otimes \Psi^j = 0.
\]

Since the carrier of \( f^k \Phi^j \) is contained in \( W^k \subset U_a \overline{W}_a^k \) we can apply Lemma III with

\[
U_a = V_a^k \quad \text{and} \quad A_a = \overline{W}_a^k.
\]

We thus obtain

\[
\sum_j f^k \Phi^j \otimes \Psi^j = 0
\]

and summing over \( k \)

\[
\sum_j \Phi^j \otimes \Psi^j = 0.
\]

The above theorem is thereby proved.