NOTE ON A WHITEHEAD PRODUCT

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The aim of this note is to prove

**Theorem.** Let \( \iota_n \) generate \( \pi_n(S^n) \) and \( n=4s+1 \), \( s \) being a positive integer. Then \( [\iota_n, \iota_n] \) is not divisible by two.

Let \( i, j, \) and \( k \) be homomorphisms in the exact sequences of the fiber bundles \( O(n) \to O(n+1) \to S^n, \ n \geq 1 \)

\[
\begin{align*}
\cdots \pi_m(O(n)) & \to \pi_m(O(n + 1)) \to \pi_m(S^n) \to \pi_{m-1}(O(n)) \\
\pi_{m+1}(S^{n+1}) & \to \pi_m(O(n + 1)) \to \pi_m(S^n).
\end{align*}
\]

and \( d = j \circ k \) be the composite

\[ j \circ k \circ \iota_n \circ O(n) \to \pi_m(O(n + 1)) \to \pi_m(S^n). \]

We state an easy lemma without proof.

**Lemma 1.** Let \( (E, F) \to (B, \ast) \) be a fibration and \( \partial \) be the boundary homomorphism in its exact homotopy sequence. Then \( \partial(\alpha \circ E \beta) = \partial(\alpha) \circ \beta \) where \( \alpha \in \pi_r(B), \ E \beta \in \pi_q(S^n) \) and \( E \) is the suspension homomorphism.

**Lemma 2.** For odd \( n \), \( [\iota_n, \iota_n] \) is divisible by two if and only if it is in the image of \( d \).

**Proof.** It is known [3, p. 120] that \( d(\iota_{n+1}) = 2 \iota_n \) if \( n \) is odd. Since \( E: \pi_{2n-1}(S^n) \to \pi_{2n}(S^{n+1}) \) is onto and its kernel is generated by \( [\iota_n, \iota_n] \) and also \( E: \pi_{2n-2}(S^{n-1}) \to \pi_{2n-1}(S^n) \) is onto, we have, for any element \( E\alpha \) in \( \pi_{2n}(S^{n+1}) \)

\[ d(E\alpha) = j \circ k(E\alpha) = j \circ k(\iota_{n+1} \circ E\alpha) = 2\iota_n \circ \alpha \quad \text{by Lemma 1} \]

\[ = 2\alpha. \]

If \( [\iota_n, \iota_n] = 2\alpha \), for some \( \alpha \) in \( \pi_{2n-1}(S^n) \), then \( E\alpha \neq 0 \) and \( d(E\alpha) = 2\alpha \). That proves the lemma.

If \( n = 4s+1 \), \( S^n \) admits only a 1-field. Consequently [2], (i) there exists \( b \in \pi_{n-1}(O(n-1)) \) such that \( i(b) = k(\iota_n) \) and \( j(b) = \eta \) where \( \eta \) is the stable element in \( \pi_{n-1}(S^{n-2}) \) (we denote all suspensions of \( \eta \) by the same symbol), and (ii) \( [\iota_n, \iota_n] = E\theta \) where \( \theta \in \pi_{2n-2}(S^{n-1}) \) is not a suspension.

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Consider the generalized Hopf invariant
\[ H : \pi_{2n-2}(S^{n-2}) \to \pi_{2n-2}(S^{2n-6}) , \]
\[ H(\eta \circ \theta) = E(\eta \# \eta) \circ H(\theta), \]
\[ = \eta \circ \eta \circ H(\theta), \]
\[ = \eta \circ \eta \circ \eta \neq 0 \]
because \( \theta \) is not a suspension, \( H(\theta) \neq 0 \) and hence \( H(\theta) = \eta \). Therefore \( \eta \circ \theta \neq 0 \) and \( k(E\theta) \neq 0 \) and the exactness of (A) implies that \( E\theta = [\iota_n, \iota_n] \) is not in the image of \( d = j \circ k \). This proves the Theorem.

REFERENCES


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