In this note we prove a generalization of a theorem of A. Gleason [3, p. 60] on continuous semicharacters of semigroups contained in abelian topological groups.

**Theorem 1 (Gleason).** Let $G$ be an abelian topological group, $S$ a subsemigroup of $G$, and $t$ an element of the interior of $S$, but $-t \not\in S$. For each complex number $z$ of modulus $\leq 1$, there is a continuous multiplicative function $\chi$ on $S$ to the complex unit disk (semicharacter of $S$) with $\chi(t) = z$.

The form of this theorem credited to Gleason in [3] applies only to locally compact groups $G$ and depends somewhat on the structure of locally compact abelian groups. As observed in [3], this theorem guarantees that the harmonic analysis of semigroups with nonempty interior in $G$, which are not subgroups, does not collapse to the ordinary theory of characters of $G$.

The proof given here is based on a version of the Hahn-Banach theorem for semigroups, [1].

**Extension Theorem.** Let $S$ be an abelian semigroup, and $\omega$ a real function on $S$ subject to the conditions

1. $-\infty \leq \omega(s) < \infty$, $s \in S$.
2. $\omega(s_1 + s_2) \leq \omega(s_1) + \omega(s_2)$, $s_i \in S$.

Let $H$ be a subsemigroup of $S$ and $\phi$ a real function on $H$ such that

3. $-\infty \leq \phi(h) < \infty$, $h \in H$.
4. $\phi(h_1 + h_2) = \phi(h_1) + \phi(h_2)$, $h_i \in H$.
5. $\phi(s + h) \leq \omega(s) + \phi(h)$, whenever $s \in S$, $h \in H$, $s + h \in H$.

Then, $\phi$ can be extended to a function $\xi$ on $S$ for which

6. $-\infty \leq \xi \leq \omega$.
7. $\xi(s_1 + s_2) = \xi(s_1) + \xi(s_2)$, $s_i \in S$.

A special case, and the only one presently necessary, is that for each $t$ in $S$ there is some $\xi$ satisfying (5) and (6) with $\xi(t) = \lim_{n \to \infty} \omega(nt)/n = \omega_\infty(t)$. For we have only to set $\phi(nt) = n\omega_\infty(t)$ for $n \geq 1$ so that if $s + kt = mt$, $s \in S$, $m$, $k \geq 1$,
\[ \phi(s + kt) = m \lim_{n \to \infty} \frac{1}{n} \omega(nt) = \lim_{n \to \infty} \frac{1}{n} \omega(mnt) \]

\[ = \lim_{n \to \infty} \frac{1}{n} (\omega(ns + nkt) \leq \lim_{n \to \infty} \left[ \frac{1}{n} \omega(ns) + \frac{1}{n} \omega(nkt) \right] \]

\[ \leq \omega(s) + k \omega_x(t) = \omega(s) + \phi(kt). \]

This special case can be derived also from Beurling's formula for the spectral radius in a commutative Banach algebra, [2].

**Proof of Theorem 1.** For \( s \in S \) define

\[ \omega(s) = \inf \{ m : s + mt \in S, m \text{ an integer} \}. \]

Clearly \(-\infty \leq \omega(s) \leq 0\) and \( \omega \) is subadditive as in (1), (2) since \( S + S \subseteq S \). Inasmuch as \( nt = t + (n-1)t \) and \( nt \in S + (n-1)t, \ n \geq 1, \ -n \leq \omega(nt) \leq 1 - n; \ \omega(nt) = -n \) precisely when \( 0 \in S \). In any case, \( \omega_x(t) = -1 \). Applying the Extension Theorem we obtain an additive function \( \xi \) on \( S \) such that \(-\infty \leq \xi \leq \omega \) and \( \xi(t) = -1 \).

\( \xi \) is continuous at \( t \), for if \( V \) is a symmetric neighborhood of \( 0 \) in \( G \) such that \( t + nV \subseteq S \), then \( nt + nV \subseteq (n-1)t + S \). Now \( \xi \leq \omega \leq 1 - n \) on the subset \( nt + nV \) of \( S; \ \xi \leq 1/n - 1 \) on \( t + V \). Let \( v \in V \) and observe that \( -2 = \xi(2t) = \xi(t+v) + \xi(t-v) \leq \xi(t+v) + 1/n - 1: \ \xi(t+v) \geq 1 - 1/n \). In general, when \( s \in S, v \in V, s + v \in S, t + v \in S \), we can write \( \xi(s + v) + \xi(t) = \xi(s + v + t) = \xi(s) + \xi(t + v) \). This proves that \( \xi \) is continuous on \( S \) and that \( Z \), the subset of \( S \) on which \( \xi = -\infty \), is open and closed; moreover \( S + Z \subseteq Z \).

For \( \lambda \) a complex number with \( \Re \lambda \geq 0 \) define \( \chi(s) = \exp [\lambda \xi(s)] \) if \( s \in Z \) and \( \chi(Z) = 0 \). Then \( |\chi| \leq 1 \) in \( S \) and \( \chi(t) = e^{-\lambda} \). \( \chi \) is multiplicative and continuous because \( Z \) is an open-closed ideal of \( S \). This completes the proof.

When \( G \) is locally compact and \( S \) measurable, the map \( \lambda \mapsto \exp [\lambda \xi] \) defines an analytic disk in the maximal ideals of certain Banach algebras.

**References**


**Yale University**