THE LOCAL FINITE-AREA PRINCIPLE IN THE HALF-PLANE

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1. The familiar finite-area principle of Fejér asserts that if the image of \(|z| < 1\) under the analytic mapping \(w = f(z) = \sum_{n=0}^\infty a_n z^n\) is of finite area (counting multiplicities), then \(\sum a_n z^n\) converges a.e. on \(\|z\| = 1\) and uniformly on closed arcs of continuity. This result was localized by Zygmund [3] and by Lusin [1] who considered the image of a region bounded by a simple Jordan arc in \(|z| < 1\) and an arc \(\alpha \leq \theta \leq \beta\) of \(|z| = 1\). They showed that if \(a_n = o(1)\) then the conclusions of the Fejér theorem hold relative to the arc \([\alpha, \beta]\) and, for \(a_n = o(n^k), k > -1\), convergence can be replaced by \((C, k)\) summability. It should be noted that the Tauberian conditions in this result are necessary in order that there be a point of convergence (or \((C, k)\) summability) on \(|z| = 1\).

The result we will establish is a localized finite area theorem for functions analytic in a half-plane.

**Theorem.** Let \(f(s) = \int_0^\infty e^{-s\tau} \gamma(x)\), where \(s = \sigma + i\tau\), be analytic in the half-plane \(\sigma > 0\). Suppose that

\[(A_0) \quad \alpha(x) = \sup_{0 \leq h \leq 1} \left| \gamma(x + h) - \gamma(x) \right| = o(1).\]

Let \(\Omega\) be a region in \(\sigma > 0\) bounded by a segment \([i\alpha, i\beta]\) of \(\sigma = 0\) and a Jordan arc. If

\[\int_\alpha^\beta \int |f'|^2 d\sigma d\tau < \infty,\]

then \(\int_0^\infty e^{-\sigma x} \gamma(x)\) converges a.e. on the segment \([i\alpha, i\beta]\) and uniformly on any closed subsegment of continuity. If \((A_0)\) is replaced by

\[(A_k) \quad \alpha(x) = o(x^k), \quad k > 0,\]

then convergence is replaced by \((C, k)\) summability in the conclusion.

It should be noted that \((A_0)\) is a necessary condition for convergence of the integral at one point of \(\sigma = 0\), but \((A_k)\), contrary to \([5, p. 335]\) is not necessary for \((C, k)\) summability. A counterexample is given in §3.

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2. We turn now to the proof of the theorem. We consider only the case of \((C, k)\) summability. Clearly there will be no loss of generality if we assume that \(\gamma(x) \equiv 0\) for \(0 \leq x < \delta\) and that \(0 < \alpha < \beta < 2\pi\).

Let \(\gamma_1\) and \(\gamma_2\) be the odd and even extensions of \(R(\gamma)\) and \(S(\gamma)\) to \((-\infty, \infty)\). Set \(\phi = (\gamma_1 + i\gamma_2)/2\) and \(t = -\tau\). Proceeding formally we have

\[
\int_0^\infty e^{-itx} d\gamma(x) = \int_{-\infty}^\infty e^{itx} d\phi(x) + i \int_{-\infty}^\infty e^{itx} (-i \text{sign } x) d\phi(x).
\]

Clearly \(\phi\) satisfies condition \((A_k)\) and, by an integration by parts, it may be seen that

\[
\psi(x) = \int_0^x y d\phi(y)
\]

satisfies condition \((A_{k+1})\).

Let \(h\) and \(p\) denote positive integers which will be chosen as large as is needed. We define

\[
\Phi_h(x) = \int_{-\infty}^x (iy)^{-h} d\phi(y), \quad \Psi_h(x) = \int_{-\infty}^x (iy)^{-h} d\psi(y)
\]

and

\[
F(t) = \int_{-\infty}^\infty e^{itx} d\Phi_{h-1}(x), \quad F^*(t) = \int_{-\infty}^\infty e^{itx} d\Psi_h(x).
\]

Clearly \(\Psi_h = -i\Phi_{h-1}\) and so \(F^*(t) = -iF(t)\).

Let \(\lambda(t)\) be a function of period \(2\pi\) and in class \(C^p\) such that

\[
\lambda(t) = 1 \quad \text{for } \alpha \leq t \leq \beta,
\]

\[
= 0 \quad \text{for } 0 < t < a < \alpha \quad \text{and } \beta < b < t < 2\pi.
\]

Let us now consider the formal \(h\)th derivative of the Fourier series of \(F^*\lambda\), \(S^{(h)}(F^*\lambda) = \sum_{\pm \infty}^\infty \beta_n e^{int}\) with \(\beta_n = o(n^{k+1})\). Clearly \(S(F\lambda) = -iS(F^*\lambda)\) and so

\[
S^{(h-1)}(F\lambda) = \sum_{-\infty}^\infty b_n e^{int}
\]

with \(b_n = o(n^k)\) and \(\beta_n = -nb_n\). Let

\[
g(z) = \sum_{-\infty}^\infty b_n r^{|n|} e^{int} + i \sum_{-\infty}^\infty (-i \text{sign } n) b_n r^{|n|} e^{int} = \sum_{0}^\infty c_n z^n
\]

where \(c_n = 2b_n\), \(z = re^{it}\). Then
\begin{align*}
\sum_{-\infty}^{\infty} nb_n r^{\alpha n} e^{i \alpha \eta t} + i \sum_{-\infty}^{\infty} (-i \text{ sign } n) nb_n r^{\alpha n} e^{i \alpha \eta t} &= \sum_{0}^{\infty} 2b_n n z^n = zg'(z).
\end{align*}

Applying the method employed by Zygmund in [4, Theorem 9] to the function \( \psi \) we see that, as \( \omega \to \infty \), the differences
\begin{align*}
\int_{-\omega}^{\omega} x e^{i \alpha \eta t} d\phi(x) &= \sum_{|n| \leq \omega} \beta_n e^{i \alpha \eta t} \\
\int_{-\omega}^{\omega} x e^{i \alpha \eta t} (-i \text{ sign } x) d\phi(x) &= \sum_{|n| \leq \omega} \beta_n (-i \text{ sign } n) e^{i \alpha \eta t}
\end{align*}
are uniformly \((C, k+1)\) summable in \([\alpha, \beta]\), the first difference to zero and the second to a finite value.

We observe now that
\begin{align*}
\Re(f'(s)) &= -\int_{-\infty}^{\infty} e^{-\sigma |x|} e^{i \alpha \eta t} |x| d\phi(x) \\
\Im(f'(s)) &= \int_{-\infty}^{\infty} e^{-\sigma |x|} e^{i \alpha \eta t} x \, d\phi(x).
\end{align*}

Since the uniform Cesàro summability implies uniform Abel summability we have
\begin{align*}
i\Im(f'(s)) &= \sum_{-\infty}^{\infty} nb_n e^{i \alpha \eta t} r^n \to 0, \\
i\Re(f'(s)) &= \sum_{-\infty}^{\infty} nb_n (i \text{ sign } n) e^{i \alpha \eta t} n \to \text{ finite value}
\end{align*}
uniformly on \([\alpha, \beta]\) as \( \sigma = -\log r \to 0^+ \). Hence for some \( \epsilon > 0 \) there is an \( M > 0 \) such that
\[ |f'(s)|^2 + M \geq |zg'(z)|^2 > 1/2 \left| g'(z) \right|^2 \]
for \( \alpha \leq t \leq \beta \) and \( 0 < \sigma = -\log r < \epsilon \).

Thus there exists an \( M' > 0 \) and a region \( \Omega' \) in \( |z| < 1 \) bounded by the arc \( \alpha \leq t \leq \beta \) of \( |z| = 1 \) and a simple Jordan arc in \( |z| < 1 \) such that
\[ \int_{\Omega} \int |g'(z)|^2 r dr dt \leq M' + 2 \int_{\Omega} \int |f'(s)|^2 d\sigma d\tau < \infty. \]

Since \( c_n = o(n^k) \), the localized finite area theorem of Zygmund is applicable. Thus \( \sum_{\alpha}^{\infty} b_n e^{i \alpha \eta t} \) is \((C, k)\) summable a.e. on \((\alpha, \beta)\) and uniformly on closed subarcs of continuity. This implies the same for \( \sum_{-\infty}^{\infty} b_n e^{i \alpha \eta t} \) and \( \sum_{-\infty}^{\infty} (-i \text{ sign } n) b_n e^{i \alpha \eta t} \).
If we now apply the method of Zygmund to the function \( \phi \) satisfying condition (A\(_k\)) we find that the differences
\[
\int_{-\omega}^{\omega} e^{itx} d\phi(x) - \sum_{|n| \leq \omega} b_n e^{int},
\]
\[
\int_{-\omega}^{\omega} e^{itx} (i \text{ sign } x) d\phi(x) - \sum_{|n| \leq \omega} b_n (i \text{ sign } n) e^{int}
\]
are uniformly summable \((C, k)\) in \([\alpha, \beta]\), the first to zero and the second to a finite value. The summability properties of the integrals are then the same as those of the series, which establishes the theorem.

3. Consider now the function
\[
\gamma(x) = \begin{cases} 
0 & \text{if } n \leq x \leq n + 1 - 1/2^n, \\
2^n & \text{if } n + 1 - 1/2^n < x < n + 1 
\end{cases}
\]
for \(n = 0, 1, 2, \ldots\). Then
\[
(C, 1) \int_0^\infty d\gamma(x) = \lim (1/\omega) \int_0^\omega \int_0^u d\gamma(x) \, du
\]
\[
= \lim (1/\omega) \int_0^\omega \gamma(u) \, du = 1
\]
since
\[
[\omega] \leq \int_0^\omega \gamma(u) \, du \leq [\omega] + 1.
\]
But
\[
\sup_{0 \leq h \leq 1} |\gamma(x + h) - \gamma(x)| = 2^{[x]} \neq o(x).
\]
Hence \(\int_0^\infty e^{-sz} d\gamma(x)\) is \((C, 1)\) summable at \(s = 0\), but \(\gamma(x)\) does not satisfy condition (A\(_1\)) contrary to [5, p. 335].

REFERENCES


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