1. Introduction. Let $k$ and $K$ be algebraic number fields, $K$ a finite extension of $k$ with Galois group $G$. H. Yokoi has considered the ring of integers $O_K$ in $K$ as a $\mathbb{Z}[G]$ module (see [2]). In particular, he has proven

**Theorem.** If both $K$ and $k$ are Galois over the rationals $\mathbb{Q}$, and $G$ is cyclic of prime order, then $H^m(G, O_K) \approx H^n(G, O_K)$ for all integers $m$ and $n$.

We will prove a generalization of this. Namely

**Theorem 1.** If $G$ is a cyclic group, then $\text{ord } H^m(G, O_K) = \text{ord } H^n(G, O_K)$ for all integers $m$ and $n$.

Notice that we have dropped the hypothesis that both $k$ and $K$ be Galois over the rationals.

To see how Theorem 1 generalizes Yokoi's result, remember that if $G$ has prime order $p$, then multiplication by $p$ annihilates all the cohomology groups. Thus in this case the cohomology groups are determined up to isomorphism by their order.

The technique used to prove Theorem 1 can be used to prove other results of a similar nature. In the same situation as above let us consider $U_K$, the units of $K$, as a $\mathbb{Z}[G]$ module. Then we have

**Theorem 2.** Let $G$ be a cyclic group, and suppose that no infinite prime of $k$ is ramified in $K$. If $\text{ord } G = n$, then $n \cdot \text{ord } H^{2r}(G, U_K) = \text{ord } H^{2s+1}(G, U_K)$ for all integers $r$ and $s$.

The hypothesis about no infinite prime ramifying is satisfied, for example, when $K$ is totally real or when $n$ is odd.

2. Proofs of the theorems. The proofs of both theorems are easy consequences of the following lemma which is a direct generalization of a result of Chevalley in Herbrand quotients (see [1]). It has come to my attention that this generalization has been discovered independently by Dr. J. Smith of Michigan University.

We need some notation. From now on $G$ will be a cyclic group of order $n$, $\sigma$ a generator of $G$, and when $d \mid n$, $G(d)$ will be the unique
subgroup of $G$ having order $d$. $K(d)$ will be the cyclotomic field of $d$th roots of unity, $O(d)$ the ring of integers in $K(d)$, and $\zeta(d)$ a primitive $d$th root of unity.

When $d$ is a prime power, $p^i$, $(1-\zeta(d))$ is a prime ideal in $O(d)$, whose residue class field has $p$ elements. On the other hand, when $d$ is composite $1-\zeta(d)$ is a unit. This is seen as follows. Let $d=p_1p_2q$, where $p_1$ and $p_2$ are distinct primes. Notice that $\zeta(d)^{p_1q}=\zeta(p_2)$ and $\zeta(d)^{p_2q}=\zeta(p_1)$. This shows that $1-\zeta(d)$ divides both $1-\zeta(p_1)$ and $1-\zeta(p_2)$. Taking absolute norms, we see that the norm of $1-\zeta(d)$ divides both $p_1$ and $p_2$. Thus the norm of $1-\zeta(d)$ is a unit, and consequently $1-\zeta(d)$ is a unit.

Let $A$ be a finitely generated $\mathbb{Z}[G]$ module. The Herbrand quotient, $q(A)$, is defined to be the ratio of $\text{ord } H^0(G, A)$ to $\text{ord } H^1(G, A)$.

Let $A^G(d)$ be the subset of $A$ left fixed by $G(d)$, and define $r(d)$ to be the $\mathbb{Z}$ rank of $A^G(d)$.

Lemma. Let $n = \prod_p p^{i(p)}$ be the prime decomposition of $n$. Then $q(A) = \prod_{p\mid n} p^{s(p)}$ where

$$s(p) = \text{ord } H^0(G, A) - \sum_{i=1}^{i(p)} \phi(p^i)^{-1}(r(n/p^i) - r(n/p^{i-1})).$$

Proof. Notice to begin with that $Q(G) \approx Q[x]/(x^n - 1)$ where $x$ is an indeterminate. We have $x^n - 1 = \prod_{d\mid n} \Phi_d(x)$ where $\Phi_d(x)$ is the cyclotomic polynomial of $d$th roots of unity. Consequently, $Q[G] \approx \sum_{d\mid n} K(d)$. Each $K(d)$ becomes an irreducible $Q[G]$ module, where $\sigma$ acts as multiplication by $\zeta(d)$.

Consider $V = Q \otimes A$. $V$ is a $Q[G]$ module. Thus $V \approx \sum_{d\mid n} a(d)K(d)$ where the $a(d)$ are certain nonnegative integers. We easily deduce the existence of a $\mathbb{Z}[G]$ submodule $B$ of $A$ such that $A/B$ is finite, and $B \approx \sum_{d\mid n} a(d)O(d)$. From the well known properties of the Herbrand quotient we have

$$q(A) = q(B) = \prod_{d\mid n} q(O(d))^{a(d)}.$$

We now compute $q(O(d))$. For $d=1$, $O(d) = \mathbb{Z}$ acted on trivially by $G$. Let $N = \sum_{i=0}^{n-1} \sigma^i$. Then $H^0(G, Z) = Z/NZ = Z/nZ$. Since $G$ is cyclic $H^1(G, Z) \approx H^1(G, Z) = Z_N/(1-\sigma)Z = (0)$. Thus $q(O(1)) = n$.

For $d \neq 1$ we have $O(d)^G = (0)$ since $\sigma$ acts as multiplication by $\zeta(d)$. Therefore ord $H^0(G, O(d)) = 1$. On the other hand, $O(d)_N = \{ a \in O(d) \mid Na = o \} = O(d)$, and $(1-\sigma)O(d) = (1-\zeta(d))$. Thus $H^1(G, O(d)) \approx H^1(G, O(d)) = O(d)/(1-\zeta(d))$. The remarks preceding this lemma now show that $q(O(d)) = 1$ if $d$ is composite and $q(O(d)) = p^{-1}$ if $d = p^i$ is a prime power.
Putting together the information we now have, we get that $q(A) = \prod_{p | n} p^{s(p)}$ where

$$s(p) = l(p)a(1) - \sum_{i=1}^{t(p)} a(p^i).$$

To relate the $a(d)$ with the $r(d)$ notice that the $\mathbb{Z}$ rank of $A^G(d)$ is equal to the $\mathbb{Q}$ dimension of $V^G(d)$. The group $G(n/p^i)$ is generated by $\sigma^{p^i}$. From the way that $\sigma$ acts it follows that $V^G(n/p^i) = \sum_{i=0}^{t(p)} a(p^i)K(p^i)$. Therefore, $r(n/p^i) = \sum_{i=0}^{t(p)} \phi(p^i)a(p^i)$. Solving for $a(p^i)$ we get that $a(p^i) = \phi(p^i)^{-1}(r(n/p^i) - r(n/p^{i-1})).$ This completes the proof.

Proof of Theorem 1. Since $G$ is cyclic the cohomology groups are periodic of order 2. It is thus sufficient to show that $q(\mathcal{O}_K) = 1$. If $[K:Q] = N$, then the $\mathbb{Z}$ rank of $\mathcal{O}_K^d = N/d$. Substituting this information into the formula of the lemma we see that, indeed, $q(\mathcal{O}_K) = 1$.

**Corollary.** If $G$ is cyclic of square free order then $H^n(G, \mathcal{O}_K) \approx H^m(G, \mathcal{O}_K)$ for all integers $m$ and $n$.

Proof. The restriction map gives a monomorphism of the $p$-primary component of $H^i(G, \mathcal{O}_K)$ into $H^i(G(p), \mathcal{O}_K)$. It follows that the $p$-primary components of the cohomology groups under consideration are elementary. These groups are thus determined up to isomorphism by their order.

Proof of Theorem 2. Let $K$ be an algebraic number field. Denote by $r_1(K)$ the number of real primes of $K$, and by $r_2(K)$ one half the number of complex primes. The Dirichlet Unit Theorem states that rank $(\mathbb{U}_K) = r_1(K) + r_2(K) - 1$. If $k$ is a subfield of $K$, the condition that no infinite prime ramify in $K$ means that the extension of every real place is real. This implies rank $(\mathbb{U}_K) = [K:k] \text{ rank } (\mathbb{U}_k) + [K:k] - 1$. Using the notation of the lemma, with $A = \mathbb{U}_K$, we have $r(1) = d\nu(\mathcal{A}) + d - 1$. Substituting this into the formula of the lemma we get $s(p) = -l(p)$ and thus $q(\mathbb{U}_K) = n^{-1}$. This finishes the proof.

**Bibliography**