ON THE HOMOGENEITY OF INFINITE PRODUCTS OF MANIFOLDS

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Let $M$ be the Cartesian product of a countably infinite family of compact manifolds $M_i$. If every $M_i$ has a boundary, then $M$ is homogeneous [2]. Since a manifold without boundary is homogeneous, $M$ is also homogeneous if none of the $M_i$ has a boundary. It follows that $M$ is homogeneous if none or if infinitely many of the $M_i$ have a boundary. Answering a question raised by R. D. Anderson concerning the remaining case, we prove

**Theorem.** The Cartesian product of a countably infinite family of compact manifolds of which finitely many have a boundary is not homogeneous.

Since a finite product of manifolds with boundary is, in turn, a manifold with boundary, we may assume that the product contains precisely one manifold with boundary. The fact that the product is countable is irrelevant: our proof applies to any infinite (or finite) product. We assume our manifolds to be Hausdorff spaces. A space is homogeneous if for any pair of points there is a homeomorphism of the space onto itself carrying one of the points into the other. A space $X$ is essential [1, p. 519] if it cannot be deformed into a proper subset, i.e., if any homotopy $h_t: X \rightarrow X$ with $h_0 = 1$ necessarily satisfies $h_1(X) = X$; here and throughout the paper, 1 stands for the identity map.

**Lemma 1.** Let $X$ be the Cartesian product of a countable family of compact Hausdorff spaces $X_i$. If every finite product of spaces $X_i$ is essential, then $X$ itself is essential.

**Proof.** Suppose the contrary and let $h_t: X \rightarrow X$ satisfy $h_0 = 1$, $h_1(X) = C \neq X$. Then $U = X - C$ is a nonvoid open subset of $X$ and, as such, contains a subset of the form $a_1 \times \cdots \times a_n \times X_{n+1} \times X_{n+2} \times \cdots$ for a sufficiently large $n \geq 1$ and suitable points $a_i \in X_i$, $1 \leq i \leq n$. Select points $b_i \in X_i$ for $i \geq n+1$, and consider the composite

$$H_t: Y \rightarrow X \rightarrow X \rightarrow Y$$

where $Y = \prod_{1 \leq i \leq n} X_i$, $j$ embeds $Y$ as the subset $Y \times b_{n+1} \times b_{n+2} \times \cdots$.

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of $X$, and $p$ projects onto the first $n$ factors. Then, $H_0 = 1$ and $H_1(Y) \subseteq p(C)$. But $a_1 \times \cdots \times a_n \notin p(C)$; for, the contrary yields a point $c = a_1 \times \cdots \times a_n \times c_{n+1} \times c_{n+2} \times \cdots \in C$ so that $c \in U$, and this is impossible since $C \cap U = \emptyset$. Thus, $Y$ fails to be essential and the result is proved.

A point $x \in X$ is unstable [1, p. 523] if for any open neighborhood $U$ of $x$ in $X$ there is a homotopy $h_t : X \to X$ with

$$h_0 = 1, \quad h_t | X - U = 1, \quad h_t(U) \subseteq U, \quad h_1(X) \neq X.$$  

A point which is not unstable is called stable [1, p. 523]. We denote by $\overline{U}$ the closure of any subset $U$. Also, $(B^n, E^n)$ stands for the pair consisting of the closed unit ball and its interior in Euclidean $n$-space.

**Lemma 2.** Let $M$ be the Cartesian product of a countable family of compact manifolds without boundary. Let $X$ be a compact Hausdorff space and suppose $x \in X$ has an open neighborhood $U$ such that the pair $\left( \overline{U}, U \right)$ is homeomorphic to $(B^n, E^n)$. Then, the point $x \times m$ is stable in $X \times M$ for any $m \in M$.

**Proof.** Let $S$ be the quotient space obtained from $X$ by shrinking to a point the subset $X - U$; let $f : X \to S$ be the identification map. Suppose $x \times m$ is unstable and let $h_t : X \times M \to X \times M$ be a homotopy satisfying the conditions (1) with respect to the neighborhood $U \times M$ of $x \times m$. It is readily seen that there exists a unique homotopy $H_t$ yielding commutativity in the diagram

$$\begin{array}{ccc}
S \times M & \xrightarrow{H_t} & S \times M \\
\uparrow f \times 1 & & \uparrow f \times 1 \\
X \times M & \xrightarrow{h_t} & X \times M
\end{array}$$

Obviously, $H_0 = 1$. There is a point $a \times b \notin h_1(X \times M)$, and it necessarily lies in $U \times M$. Hence, $f(a) \times b \notin H_1(S \times M)$ and $S \times M$ is proved to be inessential. However, $S$ is homeomorphic to an $n$-sphere, any finite product of compact manifolds without boundary is a compact manifold without boundary, and any such $g$-dimensional manifold $Q$ is essential since its Čech cohomology group $H^q(Q, Z_2) \neq 0$ whereas $H^q(P, Z_2) = 0$ for any closed proper subset $P \subseteq Q$. Thus, by Lemma 1, $S \times M$ is essential and Lemma 2 is proved.

**Lemma 3.** Let $X$ be any space and suppose $Y$ is completely regular. If $a \in X$ is unstable, then $a \times b$ is unstable in $X \times Y$ for any $b \in Y$.

**Proof.** Let $W$ be any neighborhood of $a \times b$ in $X \times Y$; we may assume that $W = U \times V$ where $U$ and $V$ are open in $X$ and $Y$. Let
$h_t: X \to X$ be a homotopy satisfying (1) with respect to $U$. Select a continuous function $r: Y \to I$ with $r(b) = 1$, $r(Y - V) = 0$. Then, $a \times b$ is readily seen to be unstable using the homotopy $H_t: X \times Y \to X \times Y$ given by $H_t(x \times y) = h_{tr(y)}(x) \times y$.

To prove the theorem, it only remains to note that an inner point of a manifold with boundary satisfies the assumption in Lemma 2, whereas a boundary point is obviously unstable. Thus, the Cartesian product discussed has both stable and unstable points, and cannot be homogeneous.

**References**


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