OPEN MANIFOLDS WITH MONOTONE UNION PROPERTY

KYUNG WHAN KWUN

A space $X$ has the monotone union property if whenever $Y$ is the union of open sets $U_1 \subset U_2 \subset \cdots \subset Y$ with each $U_i$ homeomorphic to $X$ then $Y$ is necessarily homeomorphic to $X$. Brown has shown [2] that the open $n$-cell has this property. The question arises as to which open manifolds have this property. In particular, one may ask if the open $n$-cell is, at least for high dimensions, characterized by this property among open $n$-manifolds. We show that there are many manifolds with this property. Namely we prove

**Theorem.** Let $X$ be a closed p.l. manifold of dim $n \neq 4$ and $p \in X$. Then $X - p$ has the monotone union property.

Simple examples show that contractible open manifolds or the interior of a compact manifold with 1-connected boundary do not have this property in general.

We thank P. Doyle for calling this problem to our attention.

We use $\hat{M}$, $\hat{M}$ to denote the interior and the boundary of a topological manifold $M$. We divide the proof according to cases.

1. **For $n \geq 5$ and $X$ orientable.** Let $U_1 \subset U_2 \subset \cdots$ be a sequence with each $U_i$ homeomorphic to $X - p$. We suppose that $X$ is orientable and show that $U = \bigcup_i U_i$ is homeomorphic to $X - p$. Let $A$ be a compact submanifold of $X$ such that $[X - A]$ is a ball neighborhood of $p$ in $X$.

Let, for each $i = 1, 2, \cdots$, $C_1 \subset C_2 \subset \cdots$ be a sequence of compact sets such that the union is $U_i$. Define $C_j = C_1 \cup C_2 \cup \cdots \cup C_j$. Then $C_j \subset U_j$ and $\bigcup_j C_j = U$.

Let $h_i$ be homeomorphisms of $X - p$ onto $U_i$ and $A_i = h_i(A)$. By an inductive choice of $h_i$, we suppose that $\hat{A}_i \supset C_i \cup A_{i-1}$ and therefore $\bigcup A_i = U$. Now we show that $\hat{A}_i - \hat{A}_{i-1}$ is homeomorphic to $S^{n-1} \times [0, 1)$ for each $i$. Now fix $i$ and view $U_i$ as having the p.l. structure induced by $h_i$ and $X - p$. Let $W = A_i - \hat{A}_{i-1}$.

$$\pi_1 W = 1.$$ To see this, consider $\pi_1 A_{i-1} = \pi_1 A_i = \pi_1 A_{i-1} * \pi_1 W$.

By Grusko's theorem [3], $\pi_1 W = 1$.

$$A_i \subset W$$ is a homotopy equivalence.

Received by the editors March 30, 1966.

1091
Since $W$ and $\hat{A}_i$ are 1-connected by 1.1, by Whitehead's theorem, it suffices to show that it induces homology isomorphisms or $H_\ast(W, \hat{A}_i) = 0$. This is equivalent to showing $H_\ast(W, A_{i-1}) = 0$ by the relative Poincaré duality. Then by excision, we need only show $H_\ast(A_i, A_{i-1}) = 0$, or $A_{i-1} \subset A_i$ induces homology isomorphisms. Consider a part of the Mayer-Vietoris sequence for the triad $(A_i; A_{i-1}, W)$, namely

$$0 \to H_{n-1}(A_{i-1}) \xrightarrow{\phi} H_{n-1}(A_{i-1}) \oplus H_{n-1}(W) \xrightarrow{\Psi} H_{n-1}(A_i) \to 0.$$  

Since $\phi(H_{n-1}(\hat{A}_{i-1})) \subset H_{n-1}(W)$ as $\hat{A}_{i-1}$ bounds $A_{i-1}$,

$$H_{n-1}(A_{i-1}) \oplus (H_{n-1}(W)/\text{Im } \phi) \cong H_{n-1}(A_i).$$  

Since $H_{n-1}(A_{i-1}) \cong H_{n-1}(A_i)$ is free abelian, $H_{n-1}(W)/\text{Im } \phi = 0$ and $A_{i-1} \subset A_i$ induces an isomorphism $H_{n-1}(A_{i-1}) \cong H_{n-1}(A_i)$. That $A_{i-1} \subset A_i$ induces isomorphisms for other dimensions is simple.

(1.3) $A_i - A_{i-1}$ is homeomorphic to $S^{n-1} \times [0, 1]$.

Now $A_i - A_{i-1}$ is a p.l. submanifold of $U_i = h_i(X - p)$. Furthermore, $\hat{A}_i$ is a p.l. sphere. Attach a combinatorial $n$-ball $B$ along $\hat{A}_i$ to obtain a p.l. manifold $V = B \cup (A_i - A_{i-1})$. Then $H_\ast(V, B) \cong H_\ast(A_i - A_{i-1}, \hat{A}_i) = H_\ast(W, \hat{A}_i) = 0$ by 1.2. Hence $B \subset V$ is a homotopy equivalence or $V$ is contractible. Since $V$ is 1-connected at infinity, by [5], $V$ is $n$-space. Now by the generalized Schoenflies theorem [1], $A_i - A_{i-1}$ is homeomorphic to $S^{n-1} \times [0, 1]$.

(1.4) $\hat{A}_i - \hat{A}_{i-1}$ is homeomorphic to $S^{n-1} \times [0, 1]$.

This follows from (1.3) by a simple argument.

(1.5) $U$ is homeomorphic to $X - p$.

This can be directly proved. But in order to avoid epsilonics, we use [2]. Replace $A_1$ by $n$-cell. So that the situation is the monotone union of open $n$-cells. In this case $U - \hat{A}_1$ is homeomorphic to $S^{n-1} \times [0, 1]$. So that $\hat{A}_1$ is homeomorphic to $U$.

2. For $n \geq 5$ and $X$ nonorientable. Now suppose $X$ is nonorientable. Then so is $X - p$. Consider the similar $U_i, C_i, A_i$. Consider the inclusion $A_{i-1} \subset A_i$. Let $p : B_i \rightarrow A_i$ be the 2-sheeted orientable covering of $A_i$. Let $S_i, S'_i, S_{i-1}$ and $S'_{i-1}$ be the disjoint $(n-1)$-spheres in $B_i$ such that $p^{-1}(\hat{A}_i) = S_i \cup S'_i$. One argues that $B_{i-1} = p^{-1}(A_{i-1})$ has two complementary domains $P$ and $P'$ in $B_i$ such that $P \supset S_i, P' \supset S'_i$. For instance, as before $\pi_1(A_i - A_{i-1}) = 1$ by Van Kampen's theorem and Grusko's theorem, and $p^{-1}(A_i - \hat{A}_{i-1})$ is the disjoint union of two
homeomorphic copies. Let $W = \overline{P}$, $W' = \overline{P}'$. Once we show that $S_{i-1} \subset W$ (therefore also $S'_{i-1} \subset W'$) induces homology isomorphisms, the result can be proved along a line similar to the orientable case. Consider the exact sequence

$$0 \rightarrow H_{n-1}(S_{i-1} \cup S'_{i-1}) \xrightarrow{\Phi} H_{n-1}(B_{i-1}) \oplus H_{n-1}(W) \oplus H_{n-1}(W') \xrightarrow{\Psi} H_{n-1}(B_i) \rightarrow 0$$

which is a part of the Mayer-Vietoris sequence for the triad $(B_i; B_{i-1}, W \cup W')$. Every group appearing above is a finitely generated free abelian group as $B_i$ is compact and orientable. Hence $H_{n-1}(W) \approx H_{n-1}(W') \approx \mathbb{Z}$. Let $b \in H_{n-1}(B_{i-1})$ be such that $(b, 0, 0) \in \text{Ker } \Psi = \text{Im } \Phi$. Since $H_{n-1}(S_{i-1}) \rightarrow H_{n-1}(W') = \mathbb{Z}$ and $H_{n-1}(S_{i-1}) \rightarrow H_{n-1}(W) \approx \mathbb{Z}$ are nontrivial, $b = 0$. On the other hand, cokernel $[H_{n-1}(B_{i-1}) \rightarrow H_{n-1}(B_i)]$ is isomorphic to a torsion subgroup of $H_{n-1}(B_i; B_{i-1}) \approx H_{n-1}(W, S_{i-1}) \oplus H_{n-1}(W', S'_{i-1})$ which is free. Hence $H_{n-1}(B_{i-1}) \rightarrow H_{n-1}(B_i)$ is an isomorphism. This completes the proof except for homology isomorphisms at dimensions $\neq n-1$. But this is again easy to show.

3. For $n \leq 3$. We only consider the case in which $X$ is orientable. For nonorientable case one again passes to the orientable covering. Now the case $n = 2$ is trivial, so we suppose $X$ is orientable and of dimension 3. Let $A_1, A_2$ be compact manifolds homeomorphic to $A$ such that $A_1 \subset A_2$. It suffices to show that $A_2 - A_1$ is homeomorphic to $S^2 \times [0, 1]$. Let $M_2$ is a manifold obtained from $A_2$ by attaching 3-cell, $M_1$ obtained from $A_1$ by attaching a 3-cell. Let $M_3$ be a manifold obtained from $A_2 - A_1$ by attaching two disjoint 3-cells. Then $M_2 = M_1 \# M_3$. By [4], $M_3$ is a 3-sphere so that $A_2 - A_1$ is homeomorphic to $S^2 \times [0, 1]$.

References


Michigan State University