CHARACTERISTIC ROOTS OF $M$-MATRICES

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A square matrix $B$ is called a nonnegative matrix (written $B \geq 0$) if each element of $B$ is a nonnegative number. It is well known [1] that every nonnegative matrix $B$ has a nonnegative characteristic root $p(B)$ (the Perron root of $B$) such that each characteristic root $\lambda$ of $B$ satisfies $|\lambda| \leq p(B)$.

A square matrix $A$ is called an $M$-matrix if it has the form $k \cdot I - B$, where $B$ is a nonnegative matrix, $k > p(B)$, and $I$ denotes the identity matrix. In case $A$ is a real, square matrix with nonpositive off-diagonal elements, each of the following is a necessary and sufficient condition for $A$ to be an $M$-matrix [4, p. 387].

1. Each principal minor of $A$ is positive.
2. $A$ is nonsingular, and $A^{-1} \geq 0$.
3. Each real characteristic root of $A$ is positive.
4. There is a row vector $x$ with positive entries ($x > 0$) such that $xA > 0$.

If $A$ is an $M$-matrix, then $A$ has a positive characteristic root $q(A)$ which is minimal, in the sense that for each characteristic root $\beta$ of $A$, $q(A) \leq |\beta|$ [4, p. 389]. Bounds for $q(A)$ can be readily obtained using known bounds for the Perron root of a nonnegative matrix. In this paper, as in [2], we reverse this procedure, studying the characteristic roots of $M$-matrices in order to find new bounds for Perron roots.

Ky Fan proved the following lemma [3]:

**Lemma A.** Let $A = (a_{ij})$ be an $M$-matrix of order $n$. Then the matrix $C = (c_{ij})$ given by

$$c_{ij} = a_{ij} - a_{in}a_{nj}(1/a_{nn}) \quad (i, j = 1, 2, \ldots, n - 1).$$

is an $M$-matrix, and $c_{ij} \leq a_{ij} \ (i, j = 1, 2, \ldots, n-1)$.

In [2] we generalized Lemma A by using certain principal minors in place of single elements of $A$. Here we prove a different generalization of Fan's lemma, contained in Theorem 1.

We denote the submatrix of a matrix $A$ formed using rows $i_1, i_2, \ldots, i_p$ and columns $j_1, j_2, \ldots, j_p$ by $A(i_1, i_2, \ldots, i_p; j_1, j_2, \ldots, j_p)$. For principal submatrices we abbreviate this to $A(i_1, i_2, \ldots, i_p). A_{i,j}$ denotes the element of $A$ in row $i$ and column $j$.

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**Theorem 1.** Let $M$ be an $M$-matrix of order $n = mk$, partitioned into the form

$$M = \begin{bmatrix}
A_{11} & A_{12} & \cdots & A_{1k} \\
A_{21} & A_{22} & \cdots & A_{2k} \\
\vdots & \vdots & \ddots & \vdots \\
A_{k1} & A_{k2} & \cdots & A_{kk}
\end{bmatrix},$$

where each $A_{ij}$ is an $m \times m$ matrix. Let $\phi(M)$ denote the matrix

$$\begin{bmatrix}
B_{11} & B_{12} & \cdots & B_{1,k-1} \\
B_{21} & B_{22} & \cdots & B_{2,k-1} \\
\vdots & \vdots & \ddots & \vdots \\
B_{k-1,1} & B_{k-1,2} & \cdots & B_{k-1,k-1}
\end{bmatrix},$$

where $B_{ij} = A_{ij} - A_{ik}A_{kk}^{-1}A_{kj}$. Then

1. $\phi(M)$ is an $M$-matrix, of order $m(k - 1)$,
2. $\phi(M)_{i,j} \leq M_{i,j}$, for $i, j = 1, 2, \ldots, m(k - 1)$,
3. $\det \phi(M) = \det M / \det A_{kk}$, and
4. $q(\phi(M)) \geq q(M)$.

The proof of the theorem depends on two lemmas.

**Lemma 1.** Let $T = (t_{ij})$ be an $M$-matrix of order $n$, with minimal characteristic root $q(T)$. Let $\lambda$ be a number such that $\lambda < q(T)$. Then

$$\det (T - \lambda I) > 0.$$

**Proof.** If $\omega_1, \omega_2, \ldots, \omega_n$ are the characteristic roots of $T$, $\det(T - \lambda I) = \prod_{i=1}^{n} (\omega_i - \lambda)$. For each real $\omega_i$, $\omega_i \geq q(T) > \lambda$, so that $\omega_i - \lambda > 0$. The complex factors $\omega_i - \lambda$ occur in conjugate pairs whose product is positive. Thus $\det (T - \lambda I) > 0$.

**Lemma 2.** Let $S = (s_{ij})$ be an $M$-matrix of order $n$, let $\lambda$ be a number such that $0 < \lambda < q(S)$, and let $t$ be an integer such that $0 \leq t < n$. Then the determinant of the matrix obtained from $S$ by subtracting $\lambda$ from each of the first $t$ main diagonal elements $s_{11}, s_{22}, \ldots, s_{tt}$ is positive.

**Proof.** Let $T$ be the matrix obtained from $S$ by adding $\lambda$ to the main diagonal elements $s_{11}, s_{22}, \ldots, s_{tt}$. Since $\lambda > 0$, $T \geq S$ and the off-diagonal elements of $T$ are nonpositive, so $[4, p. 389]$ $T$ is an $M$-matrix and $q(T) \geq q(S)$. Thus, since $\lambda < q(T)$, Lemma 1 implies that $\det(T - \lambda I) > 0$, which proves Lemma 2.

**Proof of Theorem 1.** Since $M$ has nonpositive off-diagonal elements, $A_{ik}$ and $A_{kj}$ are nonpositive matrices if $i, j \neq k$. Also $A_{kk}$, a principal submatrix of $M$, is itself an $M$-matrix [4, p. 390], so that
$A_{ik}^{-1} \geq 0$. Thus $A_{ik} A_{kk}^{-1} A_{kj} \geq 0$, so that $B_{ij} \leq A_{ij}$, verifying (6). So $\phi(M)$ has nonpositive off-diagonal elements, and to show that $\phi(M)$ is an $M$-matrix we prove now that each principal minor of $\phi(M)$ is positive.

Let $F$ be the nonnegative, $n \times n$ matrix

$$F = \begin{bmatrix} I & 0 \\ 0 & A_{kk}^{-1} \end{bmatrix}.$$  

Then

$$MF = \begin{bmatrix} A_{11} & \cdots & A_{1k} A_{kk}^{-1} \\ A_{21} & \cdots & A_{2k} A_{kk}^{-1} \\ \cdots & \cdots & \cdots \\ A_{k1} & \cdots & I \end{bmatrix}$$  

has nonpositive off-diagonal elements. Also, by (4), since $M$ is an $M$-matrix, there is a vector $x$ with positive entries ($x > 0$) such that $xM > 0$. Then also $(xF)F > 0$, since $F \geq 0$ and $F$ has at least one positive element in each column. (Each main diagonal element of the inverse of an $M$-matrix, being the quotient of two positive principal minors, is positive.) Thus, by (4), $MF$ is an $M$-matrix. Moreover, $\phi(M) = \phi(MF)$, so it suffices to show that each principal minor of $\phi(MF)$ is positive.

To this end, write

$$MF = \begin{bmatrix} C_{11} & C_{12} & \cdots & C_{1k} \\ C_{21} & C_{22} & \cdots & C_{2k} \\ \cdots & \cdots & \cdots & \cdots \\ C_{k1} & C_{k2} & \cdots & I \end{bmatrix},$$  

with $C_{ij} = A_{ij}$ if $j \neq k$, and $C_{ik} = A_{ik} A_{kk}^{-1}$. Then it is easily verified that a typical element of $\phi(MF)$, say $(C_{ij} - C_{ik} C_{kj})_{a,b}$, can be written as the determinant

$$\det \begin{bmatrix} (C_{ij})_{a,b} & (C_{ik})_{a,1} & (C_{ik})_{a,2} & \cdots & (C_{ik})_{a,m} \\ (C_{kj})_{1,b} \\ \cdots \\ (C_{kj})_{2,b} \\ \cdots \\ (C_{kj})_{m,b} \end{bmatrix}.$$  

Thus by Sylvester's identity [5, p. 16], if $\phi(MF)(i_1, i_2, \cdots, i_l)$ is a principal submatrix of $\phi(MF)$, then
\[ \det \left\{ \phi(MF)(i_1, i_2, \ldots, i_t) \right\} = (\det I)^{t-1} \det (MF)(i_1, i_2, \ldots, i_t, n - m + 1, \ldots, n) > 0. \]

This completes the proof that \( \phi(M) \) is an \( M \)-matrix.

To verify (7), we use Sylvester’s identity again to write
\[ \det (\phi(M)) = \det \phi(MF) = \det MF = \det M \det F = \det M / \det A_{kk}. \]

Finally, to show that \( q(\phi(M)) \geq q(M) \), we write the characteristic polynomial of \( \phi(M) \) in the form
\[ \det (\phi(M) - xI) = \det \begin{bmatrix} D_{11} & \cdots & D_{1,k-1} \\ \vdots & \ddots & \vdots \\ D_{k-1,1} & \cdots & D_{k-1,k-1} \end{bmatrix}, \]
where \( D_{ij} = (A_{ij} - \delta_{ij}x) - A_{ik}A_{kj}^{-1}A_{kj} \). (\( \delta_{ij} \) is the Kronecker symbol.) Then clearly \( \phi(M) - xI \) is the same as the matrix \( \phi(Q(x)) \), where \( Q(x) \) agrees with the matrix \( M \) except that the matrix \( xI \) has been subtracted from each of the submatrices \( A_{11}, A_{22}, \ldots, A_{k-1,k-1} \). (The construction of \( \phi(Q(x)) \) from \( Q(x) \) and part (7) of Theorem 1, require only that \( A_{kk}^{-1} \) exists, and not that \( Q(x) \) be an \( M \)-matrix.) So as in (7), we have \( \det (\phi(M) - xI) = \det \phi(Q(x)) = \det Q(x) / \det A_{kk} \).

Now if \( 0 < \lambda < q(M) \), Lemma 2 implies that \( \det Q(\lambda) > 0 \). Also, \( \det A_{kk} > 0 \), since it is a principal minor of \( M \). Thus \( \det (\phi(M) - \lambda I) > 0 \), and so no real characteristic root of \( \phi(M) \) is less than \( q(M) \). I.e. \( q(\phi(M)) \geq q(M) \), completing our proof.

Applying Theorem 1, we obtain the following theorem about the Perron root of a nonnegative matrix.

**Theorem 2.** If \( B \) is a nonnegative matrix of order \( n = mk \), and \( h > p(B) \), the Perron root of \( B \), then
\[ D = hI - \phi(hI - B) \text{ is a nonnegative matrix of order } m(k - 1), \]
\[ D_{ij} \geq B_{ij} \text{ for } i, j = 1, 2, \ldots, m(k - 1), \text{ and} \]
\[ p(B) \leq p(D). \]

**Proof.** Let \( M = hI - B \). Then \( D_{ij} = h\delta_{ij} - \phi(M)_{ij} \geq h\delta_{ij} - M_{ij} = B_{ij} \geq 0 \), verifying (9) and (10). Also \( h - p(D) = q(\phi(M)) \geq q(M) = h - p(B) \), so that \( p(B) \geq p(D) \).

As in [2], the use of known lower bounds for the Perron root of \( D \) leads to bounds for \( p(B) \). Similarly, Theorem 1 gives new bounds for \( q(M) \), by applying known upper bounds for the minimal characteristic root of the \( M \)-matrix \( \phi(M) \).

**Added in proof.** The author has since discovered the fact that the
matrix $\phi(M)$ can also be constructed using the method in [2, Lemma 1].

**References**


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