This note contains, in particular, a short proof of the celebrated result of G. Kolettis which states that a group $G$ is determined by its Ulm invariants if $G$ is the direct sum of countable, reduced, $p$-primary groups [1]. All groups considered are commutative.

**Theorem.** Suppose that $G = \sum_{\lambda \in \Lambda} A_\lambda$ and $H = \sum_{\lambda \in \Lambda} B_\lambda$ are decompositions of $G$ and $H$ into direct sums of countable, reduced, $p$-groups. If $G$ and $H$ have the same Ulm invariants, then there exists a partition of $\Lambda$ into countable subsets $\Lambda_\mu$, $\mu \in M$, such that $G_\mu = \sum_{\lambda \in \Lambda_\mu} A_\lambda$ is isomorphic to $H_\mu = \sum_{\lambda \in \Lambda_\mu} B_\lambda$ for each $\mu \in M$.

**Proof.** The theorem is vacuous if $\Lambda$ is countable, so assume that $\Lambda$ is uncountable and let $\Omega$ be the smallest ordinal having the cardinality of the set $\Lambda$. Let $G[\beta] = \sum_{i \in I} \{x_i\}$. Since $G$ and $H$ are direct sums of countable groups and have the same Ulm invariants, there is a height-preserving isomorphism $\pi$ from $G[\beta]$ onto $H[\beta]$—an element of $G$ is said to have height $\alpha$ if it is contained in $p^\alpha G$ but not in $p^{\alpha+1} G$. Thus corresponding to the decomposition $G[\beta] = \sum_{i \in I} \{x_i\}$ is the decomposition $H[\beta] = \sum_{i \in I} \{y_i\}$ where $y_i = \pi(x_i)$. Since $|G[\beta]| = |G| = |\Lambda| = |\Omega|$, the index set $I$ can be taken as the initial segment of ordinals less than $\Omega$.

Suppose that $\gamma < \Omega$ and that for each $\beta < \gamma$ we have shown the existence of a subset $S_\beta$ of $\Lambda$ and a subset $I_\beta$ of $I$ such that the following conditions are satisfied.

1. $\sum_{\lambda \in S_\beta} A_\lambda[\beta] = \sum_{i \in I_\beta} \{x_i\}$ and $\sum_{\lambda \in S_\beta} B_\lambda[\beta] = \sum_{i \in I_\beta} \{y_i\}$.

2. $|S_\beta| = |I_\beta| \leq \kappa_0 |\beta|$.

3. $\alpha \in I_\beta$ if $\alpha < \beta$.

4. $S_\beta = \bigcup_{\alpha < \beta} S_\alpha$ and $I_\beta = \bigcup_{\alpha < \beta} I_\alpha$ if $\beta$ is a limit ordinal.

5. $S_\alpha \subseteq S_\beta$ and $I_\alpha \subseteq I_\beta$ if $\alpha < \beta$.

If $\gamma$ is a limit ordinal, we define $S_\gamma = \bigcup_{\alpha < \gamma} S_\alpha$ and $I_\gamma = \bigcup_{\alpha < \gamma} I_\alpha$ and observe that the conditions (1)–(5) remain valid for $\beta \leq \gamma$. 

Received by the editors March 28, 1966.

1 If $G$ is a countable, reduced, primary group, then $G[\beta]$ can be decomposed as $G[\beta] = \sum S_\alpha$ where the elements of $S_\alpha$ have height $\alpha$. The result then easily extends to direct sums of countable groups.
Suppose that $\gamma$ is not a limit ordinal. There is a minimal subset $S_{\gamma,1}$ of $\Lambda$ such that $\{\sum_{s_{\gamma,1}} A_{\gamma,1} \} \subseteq \sum_{s_{\gamma,1}} A_{\Lambda}$ and $\{\sum_{s_{\gamma,1}} B_{\gamma,1} \} \subseteq \sum_{s_{\gamma,1}} B_{\Lambda}$. There is a minimal subset $I_{\gamma,1}$ of $I$ such that $\sum_{s_{\gamma,1}} A_{\Lambda} \subseteq \sum_{s_{\gamma,1}} B_{\Lambda}$ and $\sum_{s_{\gamma,1}} B_{\Lambda} \subseteq \sum_{s_{\gamma,1}} I_{\gamma,1} \{y_i\}$. Then there is a minimal set $S_{\gamma,2}$ of $\Lambda$ such that $\sum_{s_{\gamma,1}} I_{\gamma,1} \{x_i\} \subseteq \sum_{s_{\gamma,1}} I_{\gamma,1} \{y_i\}$. Continuing in this way, we obtain ascending sequences $S_{\gamma,n}$ and $I_{\gamma,n}$ such that

$$\sum_{s_{\gamma,n}} A_{\Lambda} \subseteq \sum_{s_{\gamma,n}} I_{\gamma,n} \subseteq \sum_{s_{\gamma,n+1}} A_{\Lambda}$$

and

$$\sum_{s_{\gamma,n}} B_{\Lambda} \subseteq \sum_{s_{\gamma,n}} I_{\gamma,n} \subseteq \sum_{s_{\gamma,n+1}} B_{\Lambda}.$$

Note that $|S_{\gamma,n}| \leq \aleph_0 |\gamma|$ for each positive integer $n$. Set $S_{\gamma} = \bigcup S_{\gamma,n}$ and $I_{\gamma} = \bigcup I_{\gamma,n}$. Then conditions (1)–(5) hold for $\beta \leq \gamma$. Obviously we can use the scheme described above to show the existence of countably infinite sets $S_i$ and $I_i$ which satisfy conditions (1) and (2) when $\beta = 1$; hence there exist, for each $\beta < \Omega$, a subset $S_\beta$ of $\Lambda$ and a subset $I_\beta$ of $I$ satisfying conditions (1)–(5). Condition (3) implies that $I = \bigcup_{\beta < \Omega} I_\beta$ and condition (1) implies that $\Lambda = \bigcup_{\beta < \Omega} S_\beta$.

Define $\Lambda_0 = S_1$ and $\Lambda_\beta = S_{\beta+1} - S_\beta$ for $1 \leq \beta < \Omega$. We know that $\sum_{s_{\Lambda_\beta}} A_{\Lambda}$ and $\sum_{s_{\Lambda_\beta}} B_{\Lambda}$ have the same Ulm invariants, for (the restriction of) $\pi$ is a height-preserving isomorphism between their socles $\sum_{s_{\Lambda_\beta}} x_i$ and $\sum_{s_{\Lambda_\beta}} y_i$. We wish to show that $\sum_{s_{\Lambda_\beta}} A_{\Lambda}$ and $\sum_{s_{\Lambda_\beta}} B_{\Lambda}$ have the same Ulm invariants. Since

$$\sum_{s_{\lambda_1}} A_{\Lambda} = \sum_{s_{\lambda}} A_{\Lambda} + \sum_{s_{\lambda_\beta}} A_{\Lambda}$$

and

$$\sum_{s_{\lambda_1}} B_{\Lambda} = \sum_{s_{\lambda}} B_{\Lambda} + \sum_{s_{\lambda_\beta}} B_{\Lambda}$$

and since the height-preserving isomorphism $\pi$ from $\sum_{s_{\lambda_1}} A_{\Lambda} \{p\}$ onto $\sum_{s_{\lambda_1}} B_{\Lambda} \{p\}$ maps $\sum_{s_{\lambda}} A_{\Lambda} \{p\}$ onto $\sum_{s_{\lambda}} B_{\Lambda} \{p\}$, the composition mapping $\phi \pi$ is a height-preserving isomorphism from $\sum_{s_{\lambda_1}} A_{\Lambda} \{p\}$ onto $\sum_{s_{\lambda_1}} B_{\Lambda} \{p\}$ where $\phi$ is the projection of $\sum_{s_{\lambda_1}} B_{\Lambda} \{p\}$ onto $\sum_{s_{\lambda_1}} B_{\Lambda} \{p\}$. Thus $\sum_{s_{\lambda_1}} A_{\Lambda}$ and $\sum_{s_{\lambda_1}} B_{\Lambda}$ have the same Ulm invariants.

Since $|\Lambda_\beta| \leq \aleph_0 |\beta| < |\Lambda|$ for each $\beta < \Omega$, the theorem now follows by induction on $|\Lambda|$.

**Reference**


**University of Houston**