A CHARACTERIZATION OF UNIONS OF TWO STAR-SHAPED SETS

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We shall use the notation $S(x)$ to denote the star of $x$ in $M$, that is, the set of all points $y \in M$ such that the segment $xy$ is contained in $M$. The star (in $M$) of a set $A$ is defined by $S(A) = \bigcup_{x \in A \cap M} S(x)$. We use the terminology "the point $x$ sees the point $y$" to mean the closed segment $xy$ is contained in $M$.

Let $M$ be a closed set in $E_r$. Suppose there exists a line segment $S_1$ such that each triple of points $x, y, z$ in $M$ determines at least one point $p \in S_1$ such that at least two of the points $x, y, z$ see the point $p$ via $M$. Valentine [1, Problem 6.6, p. 178] has conjectured that this property characterizes $M$ as the union of at most two star-shaped sets. That this condition is necessary follows immediately by choosing $S_1$ to be any line segment which intersects the kernels of the two star-shaped sets. A further property which is enjoyed by every union of two star-shaped sets is the following.

CONDITION A. If $S_1 = \bigcup_{i=1}^{m} I_i$ where the $I_i$ are closed intervals with at most end points in common, then of the intervals $I_i$ there is at least one pair (say $I_j$ and $I_k$) such that at least two of every triple of points of $M$, see a common point of $I_j \cup I_k$ via $M$.

If it is true that of each triple of points of $M$ at least two of them see a common point of a single interval (say $I_r$), then the pair $I_r$ and $I_s$ where $I_s = I_r$ satisfies the conclusion of condition A. The reader will note that if $m = 2$, then Condition A implies Valentine's property.

Next we note that a set $M$ satisfying either Valentine's property or Condition A consists of at most two components, for if $M$ had as many as three components the selection of a point from each of the components would violate Valentine's condition. In the two component case Valentine's condition can be stated as follows. If $x$ and $y$ are in the same component of $M$, then there exists a point $p \in S_1$ such that the segments $xp$ and $yp$ are contained in $M$. But then an application of a generalization of Krasnosel'skii's theorem [1, Theorem 6.18, p. 85] tells us that each component is star-shaped. Before proceeding with the case in which $M$ is connected we prove the following lemma.

**Lemma.** Suppose $M$ is connected and $A$ is a compact subset of $S_1$ such

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that for each three distinct points \( x, y, z \) of \( M \) at least two of these points see a common point \( p \) of \( A \). Then \( M = S(A) \).

**Proof.** Let \( q \in M \). Since \( M \) is connected, there exist sequences \( \{x_n\} \) and \( \{y_n\} \) which converge to \( q \) and such that for each \( n \), the points \( x_n, y_n \) and \( q \) are distinct. The hypothesis implies that, for each \( n \), at least one of the points \( x_n \) or \( y_n \) sees a point \( p_n \in A \). Since \( A \) is compact, a subsequence \( \{p'_n\} \) of \( \{p_n\} \) can be found which converges to \( p_0 \in A \) together with a corresponding subsequence (say \( \{x'_n\} \)) of one of the sequences \( \{x_n\} \) or \( \{y_n\} \) such that \( x'_n p'_n \subset M \). Then every point of \( q p_0 \) is a limit point of points on the segments \( x'_n p'_n \), and since \( M \) is closed, \( q p_0 \subset M \). Thus \( M = S(A) \).

**Theorem.** Let \( M \) be a closed subset of \( E_r \) which satisfies Condition A with respect to a line segment \( S_1 \). Then \( M \) is the union of at most two star-shaped sets.

**Proof.** Assume that \( M \) is connected since the case where \( M \) consists of two components has already been discussed. For each positive integer \( k \), divide \( S_1 \) into \( 2^k \) closed subintervals \( I_j \) \((j = 1, 2, \ldots, 2^k)\) of equal length and with at most end points in common. Then Condition A guarantees the existence of a pair of these intervals \( I_r \) and \( I_s \) such that at least two of every triple of points of \( M \) see a common point of \( I_r \cup I_s \) via \( M \). If for any \( k \), it is possible to choose a pair \( I_r \) and \( I_s \) with \( I_r = I_s \) and still preserve this property, we do so. By the lemma, \( M = S(I_r) \cup S(I_s) \). Then for each \( k \), define \( I_r = I_k \) and \( I_s = J_k \). It follows that for each \( k \), \( M = S(I_k) \cup S(J_k) \).

For each \( k \), select points \( x_k \in I_k \) and \( y_k \in J_k \). Since \( S_1 \) is compact, we select subsequences \( \{x_n\} \) of \( \{x_k\} \) and \( \{y_n\} \) of \( \{y_k\} \) which converge to \( x_0 \in S_1 \) and \( y_0 \in S_1 \) respectively.

We now show \( M = S(x_0) \cup S(y_0) \). Let \( p \) be an arbitrary point of \( M \). By the lemma and the definitions of \( I_n \) and \( J_n \), \( p \) sees a point \( z_n \in I_n \cup J_n \) for each \( n \). Without loss of generality we may assume that \( p \) sees \( z_n \in I_n \) for infinitely many \( n \). Let \( \epsilon \) be any positive number and define \( U \) and \( V \) to be the respective intersections of \( S_1 \) with \( \epsilon \) and \( \epsilon/2 \) spherical neighborhoods of \( x_0 \). Since \( \{x_n\} \) converges to \( x_0 \), there exists an integer \( N_1 \) such that if \( n > N_1 \), \( x_n \in V \). We select the integer \( N_2 \) such that if \( n > N_2 \), the length of \( I_n \) is less than \( \epsilon/2 \). Then for infinitely many \( n > \max(N_1, N_2) \), \( I_n \subset U \) and \( p z_n \subset M \). Since \( p \) sees a point in every neighborhood of \( x_0 \) and \( M \) is closed, the segment \( px_0 \subset M \). Thus each point of \( M \) sees at least one of the points \( x_0 \) or \( y_0 \) via a segment in \( M \). If for every \( n \), \( I_n = J_n \), then \( x_0 = y_0 \) and \( M \) is star-shaped. If \( x_0 \neq y_0 \), then \( M \) is the union of two star-shaped sets. Thus the proof is complete.
The following example shows that the assumption that $M$ is closed cannot be deleted. Consider the union of two disjoint closed discs in the plane together with the segment joining their centers. From each disc delete all the points of a diameter not parallel to the line of centers excepting the end points and the center itself. The set described is $M$ and $S_1$ is the intersection of the line of centers with $M$. Then $M$ is not closed, satisfies Valentine's condition and Condition A, but it is not the union of two star-shaped sets.

**Reference**


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