ON AN EXAMPLE IN SECOND ORDER LINEAR
ORDINARY DIFFERENTIAL EQUATIONS

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Let \( b(t) \) be a given positive nondecreasing continuous function on
the set \( t \geq 0 \). In this note we will prove the following result:

**Theorem.** There exists a positive continuously differentiable function
\( a(t) \) such that \( a'(t) \geq b(t) \) and the differential equation

\[
(1) \quad x'' + a(t)x = 0 \quad t \geq 0 \quad \left( ' = \frac{d}{dt} \right),
\]

has at least one solution \( x = x(t) \) such that

\[
(2) \quad \limsup_{t \to \infty} |x(t)| > 0.
\]

The above theorem generalizes the examples given by Milloux [4],
Hartman [3], and Galbraith, McShane, and Parrish [2], whose
methods do not necessarily produce a function \( a(t) \) with \( a'(t) \geq b(t) \),
if \( b(t) \) is taken of sufficiently large order as \( t \to \infty \). Such examples are
of interest in regard to the converse problem, i.e., what conditions
besides \( a(t) \to \infty \) as \( t \to \infty \) need to be assumed in order to know that all
solutions of (1) satisfy \( x(t) \to 0 \) as \( t \to \infty \). The book by Cesari [1, pp.
84–86] has a good discussion of this problem. Willett, Wong, and
Meir [5] list some new results in this direction. We take the occasion
to point out that in [1, p. 86] Sansone’s sufficient condition there
reported should read, “If \( a(t) \) is positive, nondecreasing, with a con-
tinuous derivative in \([t_0, + \infty]\), if \( a'(t) \to \infty \), and \( \int^{+\infty} a^{-1}(t)dt = \infty \),
then for every solution \( x(t) \) of (1) we have \( x(t) \to 0 \) as \( t \to +\infty \).” This
corrects a misprint in [1, p. 86] (where “\( = + \infty \)” was printed
as “\( < \infty \)”).

In order to prove our main theorem, we will need the simple prop-
erties of solutions to (1) stated in the following lemma.

**Lemma.** Let \( x(t) \) be any solution of (1) for a given continuous \( a(t) \),
and let \( \mu \) and \( T \) be positive numbers such that \( a(t) \geq \mu^2 \) for all \( t \) in \([0, T]\).
Then \( x' \) has finitely many zeroes in \([0, T]\), and if \( t_0 < t_1 < \cdots < t_n \) are
those zeroes then \( 0 < t_k - t_{k-1} \leq 2\pi \mu^{-1} \) \((k = 1, 2, \cdots, n)\).

**Proof.** By the Sturm Comparison Theorem, for any solution \( x(t) \)

Presented to the Society, December 2, 1965; received by the editors January 3,
1966.
of (1), \( x(t) \) has a finite number of zeroes in the interval \( 0 \leq t \leq T \). If \( \tau_1 \) and \( \tau_2 \) are successive zeroes, then \( \tau_2 - \tau_1 \leq \pi \mu^{-1} \). Now between \( \tau_1 \) and \( \tau_2 \), \( x(t) \) is either always positive or always negative. Hence, since \( x''(t) = -a(t)x(t) \), \( x \) is either strictly concave or strictly convex for \( \tau_1 \leq t \leq \tau_2 \), and so \( x \) has exactly one critical point between \( \tau_1 \) and \( \tau_2 \). Clearly the lemma follows.

\textbf{Proof of the Theorem.} Let

\[ a_1(t) = 4\pi^2 + \int_0^t b(s)ds \]

for \( t \) in \([0, t_1]\), where \( t_1 \) is such that \( \frac{1}{2} \leq t_1 \leq 1 \) and \( x_1'(t_1) = 0 \) for \( x_1(t) \) the unique solution to

\[ x_1'' + a_1(t)x_1 = 0, \quad x_1(0) = x_0 > 0, \quad x_1'(0) = 0. \]

By the lemma, such a point \( t_1 \) must exist. For \( t > t_1 \) define \( a_1(t) = x_1(t) = 0 \). Finally, let \( 0 < \epsilon_k < 1 \) be a given sequence of numbers such that \( x_1^2(t_k) \geq (1 - \epsilon_k)x_0^2 \) and \( \sum_{n=1}^{\infty} \epsilon_n < \infty \).

The proof of the theorem is inductive in nature. Suppose that a set of numbers \( 0 = t_0 < t_1 < \cdots < t_{n-1} \) such that

\[ \frac{1}{2} \leq t_k - t_{k-1} \leq 1 \quad (k = 1, 2, \cdots, n - 1) \]

and a set of functions \( a_k(t), x_k(t) \) \((k = 1, 2, \cdots, n - 1)\) have been determined so that the following holds \((k = 1, 2, \cdots, n - 1)\):

\[ x_k'' + a_k(t)x_k = 0 \quad \text{and} \quad a_k'(t) \geq b(t) \quad \text{for} \quad t \in [t_{k-1}, t_k], \]

\( x_k(t) = a_k(t) = 0 \) for \( t \in [t_{k-1}, t_k] \),

\[ x_k(t_{k-1}) = x_{k-1}(t_{k-1}), \quad x_k'(t_{k-1}) = x_k'(t_k) = 0, \]

\[ a_k(t_{k-1}) = a_{k-1}(t_{k-1}), \quad a_k'(t_{k-1}) = b(t_{k-1}), \quad a_k'(t_k) = b(t_k). \]

Suppose also that

\[ x_k^2(t_k) \geq (1 - \epsilon_k)x_k^2(t_{k-1}) \quad (k = 1, 2, \cdots, n - 1). \]

If we can obtain by induction a sequence of points \( \{t_k\} \) and functions \( \{a_k(t)\} \) and \( \{x_k(t)\} \) satisfying (3), (4), and (5), the theorem will follow by taking

\[ a(t) = \sum_{k=1}^{\infty} a_k(t) \quad \text{and} \quad x(t) = \sum_{k=1}^{\infty} x_k(t). \]

From (5) we obtain

\[ x^2(t_k) \geq (1 - \epsilon_k)x^2(t_{k-1}) \geq \prod_{j=1}^{k} (1 - \epsilon_j)x_0^2 \quad (k = 1, 2, \cdots). \]
Since \( t_k \to \infty \) as \( k \to \infty \) and \( \sum_{j=1}^{\infty} \epsilon_j < \infty \),

\[
\limsup_{t \to \infty} x^2(t) \geq \prod_{j=1}^{\infty} (1 - \epsilon_j)x_0^2 > 0.
\]

Thus, we have to show the existence of a point \( t_n \) and functions \( a_n(t) \) and \( x_n(t) \) such that (3), (4), and (5) hold with \( k = n \). Let \( \alpha \) be any positive number satisfying \( \alpha > a_{n-1}(t_{n-1}) + b(1 + t_{n-1}) \) and \( \alpha > (\epsilon_{n-1} - 1)b(1 + t_{n-1}) \). For \( \alpha \) fixed, let \( s_n \) be any number satisfying \( 0 < s_n - t_{n-1} < \frac{1}{2} \) and

\[
s_n - t_{n-1} < \left\{ \frac{2}{\alpha} \left[ 1 - (1 - \epsilon_n)^{1/2}(1 + \alpha^{-1}b(1 + t_{n-1}))^{1/2} \right] \right\}^{1/2}.
\]

Finally, let

\[
a_n(t) = \int_{t_{n-1}}^{t} b(\tau) \, d\tau + \frac{1}{2} \left( \alpha - \int_{t_{n-1}}^{s_n} b(\tau) \, d\tau \right) \left( 1 - \cos \pi \frac{t - t_{n-1}}{s_n - t_{n-1}} \right)
+ \frac{1}{2} a_{n-1}(t_{n-1}) \left( 1 + \cos \pi \frac{t - t_{n-1}}{s_n - t_{n-1}} \right)
\]

for \( t_{n-1} \leq t \leq s_n \), and let

\[
a_n(t) = \alpha + \int_{s_n}^{t} b(\tau) \, d\tau
\]

for \( s_n \leq t \leq t_n \). Here, \( t_n \) is any point such that \( \frac{1}{2} \leq t_n - t_{n-1} \leq 1 \) and \( x_n'(t_n) = 0 \) for \( x_n(t) \) defined on \([t_{n-1}, t_n]\) to be the solution of

\[
x_n'' + a_n(t)x_n = 0, \quad x_n(t_{n-1}) = x_{n-1}(t_{n-1}), \quad x_n'(t_{n-1}) = 0.
\]

By the lemma, such a point \( t_n \) must exist. Let \( x_n(t) = a_n(t) = 0 \) for \( t \) not in \([t_{n-1}, t_n]\). It is easy to verify that \( a_n(t) \) is a continuously differentiable function on \([t_{n-1}, t_n]\), and that \( a_n(t) \) and \( x_n(t) \) satisfy (4) with \( k = n \).

We will now prove that \( x_n(t) \) satisfies (5) with \( k = n \). For the sake of brevity in what follows, let \( x = x_n \) and \( a = a_n \). Since \( x'(t_{n-1}) = 0 \), by Taylor's Theorem we obtain

\[
x(s_n) - x(t_{n-1}) = \frac{1}{2}(s_n - t_{n-1})^2 x''(c) \quad (t_{n-1} < c < s_n).
\]

Because \( a' \geq 0 \), the set of maxima of \( |x(t)| \) are decreasing; hence

\[
|x''(c)| = a(c) \cdot x(c) \leq a(s_n) \cdot x(t_{n-1})|.
\]

So

\[
x(s_n) \geq \left[ 1 - \frac{1}{2}(s_n - t_{n-1})^2 a(s_n) \right] x(t_{n-1})|.
\]
In order to estimate $|x(t_n)|$, we integrate $x'x'' + axx' = 0$ by parts to obtain

$$a(t_n)x^2(t_n) = [x'(s_n)]^2 + a(s_n)x^2(s_n) + \int_{t_n}^{t_{n+1}} a'(t)[x(t)]^2 dt.$$ 

Hence

$$x^2(t_n) \geq \frac{a(s_n)}{a(t_n)} x^2(s_n) \geq \frac{x^2(s_n)}{1 + \alpha^{-1}b(1 + t_{n-1})},$$

since $a(s_n) = \alpha$ and

$$a(t_n) - \alpha = \int_{s_n}^{t_n} b(t) dt \leq b(t_n)(t_n - s_n) \leq b(1 + t_{n-1}).$$

Combining (7) and (8), we obtain

$$x^2(t_n) \geq \frac{[1 - \frac{1}{2}(s_n - t_{n-1})^2\alpha]^2}{1 + \alpha^{-1}b(1 + t_{n-1})} x^2(t_{n-1}).$$

But from (6) it follows that

$$\frac{[1 - \frac{1}{2}(s_n - t_{n-1})^2\alpha]^2}{1 + \alpha^{-1}b(1 + t_{n-1})} > 1 - \epsilon_n.$$ 

Hence, $x^2(t_n) \geq (1 - \epsilon_n)x^2(t_{n-1})$, and the theorem follows.

References

4. H. Milloux, Sur l'équation différentielle $x'' + A(t)x = 0$, Prace Mat. 41 (1934), 39–53.

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