A NOTE ON BRANCH POINTS OF MINIMAL SURFACES

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1. Introduction and formulation of result. One of the classical formulations of the problem of Plateau is as follows:

Given a Jordan curve $\Gamma$ in $n$-dimensional Euclidean space $E^n (n \geq 3)$, determine a real vector function $X(u, v) = (x_1(u, v), \ldots, x_n(u, v))$, defined for $u^2 + v^2 \leq 1$, with the properties:

(a) $X(u, v)$ is continuous for $u^2 + v^2 \leq 1$ and harmonic for $u^2 + v^2 < 1$;
(b) $E=G$, $F=0$ in $u^2 + v^2 < 1$, where $E=X_u \cdot X_u$, $F=X_u \cdot X_v$, $G=X_v \cdot X_v$ (the subscripts denoting partial differentiation);
(c) by the equation $X = X(u, v)$ (position vector with respect to some cartesian coordinate system in $E^n$) the circle $u^2 + v^2 = 1$ is mapped topologically onto $\Gamma$.

If the conditions (a), (b), (c) are satisfied by the function $X(u, v)$, the surface $S$, given by the equation $X = X(u, v)$, $u^2 + v^2 < 1$, is called a generalized minimal surface of the type of the disc, bounded by $\Gamma$. A singular point of the surface $S$, i.e. a point where $EG - F^2 = 0$, and hence $X_u = X_v = 0$, is called a branch point of order $m$ of the minimal surface, if all the partial derivatives of $X(u, v)$ with respect to $u$ and $v$ vanish at this point up to and including the order $m \geq 1$, while at least one of the derivatives of order $m + 1$ is different from the zero vector. It is easy to see that the branch points must be isolated. The classical solutions of Plateau's problem leave the question open whether, for a given Jordan curve $\Gamma$, there is always a generalized minimal surface of the type of the disc, bounded by $\Gamma$. Therefore statements are of interest, which give information about the possibly existing branch points of any generalized minimal surface of the type of the disc, bounded by a Jordan curve $\Gamma$, as soon as certain geometric properties of $\Gamma$ are known. In this sense we have:

Theorem. Let $\Gamma$ be a Jordan curve with total curvature $\kappa(\Gamma)$ in $E^n$. If a generalized minimal surface of the type of the disc, bounded by $\Gamma$, has branch points $(u_i, v_i)$ (with $u_i^2 + v_i^2 < 1$), $1 \leq i \leq k$, of orders $m_i$ respectively, then

$$1 + \sum_{i=1}^{k} m_i \leq \frac{\kappa(\Gamma)}{2\pi}.$$
Especially, there cannot be a branch point if $\kappa(\Gamma) < 4\pi$; and there are only finitely many branch points, if $\kappa(\Gamma)$ is finite.

For an analytic Jordan curve this inequality (in a somewhat more general form) is due to Sasaki [6] (compare J. C. C. Nitsche [2], [3, p. 236]). Nitsche [3, p. 236] has asked for statements about the general case. In the following we shall use an argument of Radó [4] to prove the theorem for arbitrary Jordan curves, thus giving a partial answer to one of Nitsche’s problems on minimal surfaces (Problem 27 in [3, p. 258]). The total curvature $\kappa(\Gamma)$ of a closed curve $\Gamma$ is defined in the general case, according to Milnor [1], by $\kappa(\Gamma) = \sup \{ \kappa(\Pi) \}$, where the supremum is taken over all closed polygons inscribed in $\Gamma$, and where $\kappa(\Pi)$ for a closed polygon $\Pi$ is defined as the sum of its exterior angles. (For closed curves of class $C''$ this definition leads to the usual total curvature $\int_{\Gamma} X''(s) \, ds$ (s arc length).)

2. Proof of the theorem. Let $K = \{(u, v); u^2 + v^2 < 1\}$, let $\partial K$ be the boundary of $K$ and $\overline{K} = K \cup \partial K$. Let the nonconstant real function $h(u, v)$ be continuous in $\overline{K}$ and harmonic in $K$. A point $(u_0, v_0) \in K$ is called a critical point (of the function $h(u, v)$) of order $\geq m$, if at $(u_0, v_0)$ all the partial derivatives with respect to $u$ and $v$ vanish up to and including a certain order $m$. Let $h|\partial K$ be the restriction of the function $h(u, v)$ to $\partial K$. We want to show that $h|\partial K$ has at least \(1 + \sum_{i=1}^{k} m_i\) different relative maxima on $\partial K$, if there are $k$ different critical points $(u_i, v_i) \in K$, $1 \leq i \leq k$, of orders $\geq m_i$, respectively.

**Lemma 1.** If $(u_a, v_a) \in K$ is a critical point of order $\geq m_a$, then there are (closed) Jordan arcs $J_{a_0}, J_{a_1}, \ldots, J_{a_{2r-1}}$ with $r \geq 1 + m_a$ contained in $\overline{K}$ with the properties:

(a) $J_{a_i} \cap J_{a_j} = \{(u_a, v_a)\}$ for $i \neq j$; $(u_a, v_a)$ is an endpoint of $J_{a_i}$, $0 \leq i \leq 2r-1$;

(b) $J_{a_i} \cap \partial K = \{(u_{a_i}, v_{a_i})\}, 0 \leq i \leq 2r-1$; the points $(u_{a_0}, v_{a_0}), (u_{a_1}, v_{a_1}), \ldots, (u_{a_{2r-1}}, v_{a_{2r-1}})$ are endpoints of the corresponding arcs; they follow each other in this order if $\partial K$ is run through in positive sense;

(c) for $(u, v) \in J_{a_i} - \{(u_a, v_a)\}$ inequality $(-1)^i [h(u, v) - h(u_a, v_a)] > 0$ holds, $0 \leq i \leq 2r-1$.

The proof may easily be seen from an argument of Radó [4, p. 793].

**Lemma 2.** If the different points $(u_i, v_i) \in K$, $1 \leq i \leq k$, are critical points of orders $\geq m_i$, respectively, then there are $m \geq 1 + \sum_{i=1}^{k} m_i$ mutually disjoint, simply connected open domains $G_i, \ldots, G_m$ contained in $K$ with the properties:
(a) For \((u, v) \in K - \bigcup_{i=1}^{m} G_j\) and \((u, v) \neq (u_i, v_i), 1 \leq i \leq k\), the inequality \(h(u, v) < \max_{1 \leq i \leq k} \{h(u_i, v_i)\}\) holds;

(b) \(\partial K \cap \partial G_j\) contains in its (nonempty) relative interior (with respect to \(\partial K\)) a point where \(h|\partial K\) has a relative maximum, \(1 \leq j \leq m\).

Proof. We proceed by induction. First let \(k = 1\). Let \(J_{10}, \ldots, J_{1,2r-1}\), with \(r \geq 1 + m_1\), be the Jordan arcs belonging to the critical point \((u_1, v_1)\) according to Lemma 1, and let \((u_{10}, v_{10}), \ldots, (u_{1,2r-1}, v_{1,2r-1})\) be the corresponding endpoints on \(\partial K\). Because of properties (b) and (c) in Lemma 1, the continuous function \(h|\partial K\) must have a relative maximum between \((u_{1i}, v_{1i})\) and \((u_{1,i+2}, v_{1,i+2})\) for odd \(i\), \(1 \leq i \leq 2r - 1\) (where \((u_{1,i+2}, v_{1,i+2}) = (u_{11}, v_{11})\)). Therefore the connected components \(G_1, \ldots, G_r\) of \(K - \bigcup_{i=1}^{2r} J_i\) have the properties demanded in Lemma 2, which hence is true for \(k = 1\).

Suppose now the statement of Lemma 2 is proved for a \(k \geq 1\). Let \(K\) contain, then, the \(k + 1\) critical points \((u_i, v_i), 1 \leq i \leq k + 1\), of orders \(\geq m_i, \) respectively. After appropriate renumbering, if necessary, we may assume \(h(u_{k+1}, v_{k+1}) \geq \max_{1 \leq i \leq k} \{h(u_i, v_i)\}\). By the inductive hypothesis, to the \(k\) points \((u_i, v_i), 1 \leq i \leq k\), there belong \(s\) domains \(G_1, \ldots, G_s\), where \(s \geq 1 + \sum_{i=1}^{k} m_i\), with the properties of Lemma 2.

Because of property (a), the point \((u_{k+1}, v_{k+1})\) is contained in one of these domains, \((u_{k+1}, v_{k+1}) \in G_1\), say. Let \(J_{k+1,0}, \ldots, J_{k+1,2r-1}\), where \(r \geq 1 + m_{k+1}\), be the Jordan arcs belonging to \((u_{k+1}, v_{k+1})\) according to Lemma 1. Since for \(j\) even, \((u, v) \in J_{k+1,j} - \{(u_{k+1}, v_{k+1})\}\) implies the inequality \(h(u, v) > h(u_{k+1}, v_{k+1}) \geq \max_{1 \leq i \leq k} \{h(u_i, v_i)\}\), the arc \(J_{k+1,j}\) with \(2 \nmid j\), \(1 \leq j \leq 2r - 1\), separate the domain \(G_1\) into \(r\) simply connected domains \(G'_1, \ldots, G'_r\), each containing (except for its endpoints) one arc \(J_{k+1,j}\) with \(j\) even. Each endpoint different from \((u_{k+1}, v_{k+1})\) of such an arc is contained in a subarc of \(\partial K\) each of whose endpoints either is an endpoint of an arc \(J_{k+1,j}\), with \(2 \nmid j\), or a point of \(\partial (\partial K \cap \partial G_1)\).

Since \(h(u, v) < h(u_{k+1}, v_{k+1})\) for \((u, v) \in \partial (\partial K \cap \partial G_1)\) it is clear that the function \(h|\partial K\) must have a relative maximum in the relative interior (with respect to \(\partial K\)) of \(\partial K \cap \partial G'_i\), \(1 \leq i \leq r\). Hence the domains \(G'_1, \ldots, G'_r, G_2, \ldots, G_s\), the total number of which is \(r + (s - 1) \geq 1 + m_{k+1} + \sum_{i=1}^{k} m_i\), have the properties demanded in Lemma 2.

Thus we have proved Lemma 2 and hence the fact that \(h|\partial K\) must have at least \(1 + \sum_{i=1}^{k} m_i\) different relative maxima on \(\partial K\), if \(K\) contains \(k\) different critical points \((u_i, v_i)\) of orders \(\geq m_i\), respectively.

Now let \(\Gamma\) be a Jordan curve in \(E^n\), and let \(S\) be a generalized minimal surface of the type of the disc (in the representation given
initially), bounded by $\Gamma$. Let the different points $(u_i, v_i)$, $u_i^2 + v_i^2 < 1$, $1 \leq i \leq k$, be branch points of $S$ of orders $m_i$, respectively. Let $Y$ be a unit vector. The point $(u_i, v_i) \in K$ is a critical point of order $\geq m_i$ of the function $h(u, v) = Y \cdot X(u, v)$, which is continuous in $\overline{K}$ and harmonic in $K$. Hence, for the number $\mu(\Gamma, Y)$ (which may be $\infty$) of relative maxima of the function $Y \cdot X\big|_{\partial K}$ we have $\mu(\Gamma, Y) \geq 1 + \sum_{i=1}^{k} m_i$. Since $Y$ was arbitrary, the crookedness $\mu(\Gamma) = \min \{\mu(\Gamma, Y)\}$ (where $Y$ ranges over all unit vectors) of $\Gamma$ satisfies the inequality $\mu(\Gamma) \geq 1 + \sum_{i=1}^{k} m_i$. But according to Milnor [1, p. 253], the inequality $\kappa(\Gamma) \geq 2\pi\mu(\Gamma)$ holds; hence our theorem is proved.

References


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