ON THE EXTENSIONS OF CONTINUOUS FUNCTIONS FROM DENSE SUBSPACES

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Recently Taimanov and McDowell have been concerned with the question: when can a continuous function from a space $X$ to a compact space $Y$ be extended to a space $\alpha X$ in which $X$ is dense? An important result, e.g., obtained by Taimanov is the following

**Theorem.** Let $X$ be dense in the $T_1$-space $\alpha X$. Then in order that a continuous function $f$ from $X$ into a compact space $Y$ have a continuous extension $f^*: \alpha X \rightarrow Y$ it is necessary and sufficient that for each two disjoint closed sets $F_1$ and $F_2$, $f^{-1}[F_1]$ and $f^{-1}[F_2]$ have disjoint closures in $\alpha X$.

The following theorems provide information about the possibility of extending functions into spaces which are realcompact instead of compact.

**Theorem A.** Let $\alpha X$ be a $T_1$-space in which $X$ is dense, and let $f$ be a continuous function from $X$ into a realcompact space $Y$. Then $f$ has a continuous extension $f^*: \alpha X \rightarrow Y$ if and only if for every countable family $\{F_n\}$ of closed sets in $Y$ such that $\bigcap_n F_n = \emptyset$, $\bigcap_n \text{cl}_{\alpha X} f^{-1}[F_n] = \emptyset$.

**Proof. Sufficiency.** We suppose some $f: X \rightarrow Y$ has no extension to all of $\alpha X$. Now $Y$, being realcompact, is embeddable as a closed copy in $R^m$ for some cardinal $m$ ($R$ denotes the reals; see [3, p. 160]) and we look on $Y$ as being this subset of $R^m$. Consider the space $(R^m)^* = (R \cup \{\infty\})^m$ where $R \cup \{\infty\}$ is the one-point compactification of $R$. Then $(R^m)^*$ is compact. Then $f: X \rightarrow (R^m)^*$ also and this function does admit an extension $f^*: \alpha X \rightarrow (R^m)^*$. To show this we use Taimanov's result quoted above: Let $F_1$ and $F_2$ be disjoint closed subsets of $(R^m)^*$ and set $F'_1 = F_1 \cap R^m$ and $F'_2 = F_2 \cap R^m$. Then $F_1$, $F_2$ are disjoint in $R^m$ and by hypothesis, $\text{cl}_{\alpha X} f^{-1}[F'_1] \cap \text{cl}_{\alpha X} f^{-1}[F'_2] = \emptyset$. But $f^{-1}[F_i] = f^{-1}[F'_i], i = 1, 2$ since $f$ maps into $R^m$. Thus there exists a continuous extension $f^*: \alpha X \rightarrow (R^m)^*$. But then for some $p_0 \in \alpha X - X$, $f^*(p_0)$ has $\infty$ for at least one coordinate. Consider now the sets $F_n = (R - (-n, n))^m$, closed in the product $R^m$. Then $\bigcap_n F_n = \emptyset$ but $p_0 \in \bigcap_n \text{cl}_{\alpha X} f^{-1}[F_n]$.

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NECESSITY. Suppose \( f: X \to Y \) has an extension \( f^*: \alpha X \to Y \). Let \( \{ F_n \} \) be a family of closed sets with \( \bigcap_n F_n = \emptyset \) and suppose that there is a \( p_0 \) in \( \bigcap_n \text{cl}_x f^{-1}[F_n] \). Then
\[
p_0 \in \bigcap_n \text{cl}_x f^{-1}[F_n] \subseteq \bigcap_n f^{-1}[\overline{F_n}] = \bigcap_n f^{-1}[F_n] = f^{-1}[\bigcap_n F_n] = \emptyset.
\]

THEOREM B. Let \( \alpha X \) be a \( T_1 \)-space in which \( X \) is dense and let \( f \) be a continuous function from \( X \) into a realcompact space \( Y \). Then \( f \) has a continuous extension \( f^*: \alpha X \to Y \) if and only if for any countable discrete family \( \{ F_n \} \) of closed sets in \( Y \), \( \{ f^{-1}[F_n] \} \) is discrete in \( \alpha X \).

PROOF. SUFFICIENCY. Since \( Y \) is realcompact, \( Y \) is contained as a closed copy in \( R^m \) for some cardinal \( m \), and we consider \( Y \) to be this subset of \( R^m \). Suppose for some \( f: X \to Y \) there exists no continuous extension \( f^*: \alpha X \to Y \). As in the proof of Theorem A, \( f \), considered as a function into \( (R^m)^* = (R^\cup \{ \infty \})^m \), has an extension \( f^*: \alpha X \to (R^m)^* \). Then for some \( p_0 \in \alpha X - X \), \( f^*(p_0) \) has \( \infty \) for at least one of its coordinates in \( (R^m)^* \). Consider the sets \( I_n = [-n-1, -n] \cup [n, n+1] \) in \( R \) and the closed sets \( F_n = I_n^m \) in \( R^m \). Then every neighbourhood of \( p_0 \) in \( \alpha X \) must intersect infinitely many of the sets \( \{ f^{-1}[F_n] \} \) and also it is true that every neighbourhood of \( p_0 \) in \( \alpha X \) must intersect infinitely many of the sets \( \{ f^{-1}[F_n] \} \) for \( n \) odd or for \( n \) even. Suppose the first. Then since \( \{ F_n : n \text{ odd} \} \) is discrete in \( R^m \), the family \( \{ f^{-1}[F_n] : n \text{ odd} \} \) is discrete in \( \alpha X \), a contradiction.

NECESSITY. Suppose \( f: X \to Y \) has an extension \( f^*: \alpha X \to Y \) and consider any countable discrete family \( \{ F_n \} \) of closed sets in \( Y \). Let \( p_0 \in \alpha X \) and let \( y_0 = f^*(p_0) \in Y \). Then \( y_0 \) has a neighbourhood \( U \) that intersects at most one \( F_n \). Now, if \( (f^*)^{-1}[U] \), which is a neighbourhood of \( p_0 \) in \( \alpha X \), intersected \( \text{cl}_x f^{-1}[F_1] \) and \( \text{cl}_x f^{-1}[F_2] \), say, then \( f^*[(f^*)^{-1}[U]] = U \) would intersect \( f^*[\text{cl}_x f^{-1}[F_1]] = F_1 \) and also \( f^*[\text{cl}_x f^{-1}[F_2]] = F_2 \) since \( F_1 \) and \( F_2 \) are closed. This is a contradiction.

THEOREM C. Let \( \alpha X \) be a \( T_1 \)-space in which \( X \) is dense, and let \( f \) be a continuous function from \( X \) into a realcompact space \( Y \). Then \( f \) has a continuous extension \( f^*: \alpha X \to Y \) if and only if

(i) for any two disjoint closed subsets \( F_1 \) and \( F_2 \),
\[
\text{cl}_x f^{-1}[F_1] \cap \text{cl}_x f^{-1}[F_2] = \emptyset;
\]

(ii) for any countable decreasing set of closed subsets \( \{ F_n \} \) such that
\[
\bigcap_n F_n = \emptyset, \quad \bigcap_n \text{cl}_x f^{-1}[F_n] = \emptyset.
\]

PROOF. NECESSITY. Let \( f: X \to Y \) have a continuous extension \( f^*: \alpha X \to Y \).
(i) If for two closed $F_1, F_2$ in $Y$, $p_0 \in \text{cl}_{\alpha X} f^{-1}[F_1] \cap \text{cl}_{\alpha X} f^{-1}[F_2]$, 
\[
f^*(p_0) \in f^*[\text{cl}_{\alpha X} f^{-1}[F_1]] \cap f^*[\text{cl}_{\alpha X} f^{-1}[F_2]] \\
\subseteq \text{cl}_Y f^*[f^{-1}[F_1]] \cap \text{cl}_Y f^*[f^{-1}[F_2]] \\
= \text{cl}_Y f[f^{-1}[F_1]] \cap \text{cl}_Y f[f^{-1}[F_2]] \\
= \text{cl}_Y F_1 \cap \text{cl}_Y F_2 \\
= F_1 \cap F_2.
\]

(ii) Let $\{F_n\}$ be decreasing, closed and such that $\bigcap_n F_n = \emptyset$. Then if $p_0 \in \bigcap_n \text{cl}_{\alpha X} f^{-1}[F_n]$, 
\[
p_0 \in \bigcap_n \text{cl}_{\alpha X} (f^*)^{-1}[F_n] \subseteq \bigcap_n (f^*)^{-1}[-F] \\
= \bigcap_n (f^*)^{-1}[F_n] = (f^*)^{-1} [-\bigcap_n F] = \emptyset.
\]

**Sufficiency.** Suppose some $f: X \to Y$ has no extension to $\alpha X$. Again, $Y \subseteq \text{cl}_1 R^n$ for some cardinal $m$ and $f$ considered as a function into $(R^n)^*$ (as in Theorems A and B) has an extension $f^*: \alpha X \to (R^n)^*$. Then there must be a $p_0 \in \alpha X - X$ such that $f^*(p_0)$ has at least one coordinate equal to $\infty$. As before, we consider the closed sets $F_n = (R - (-n, n))^m$. These are decreasing and $\bigcap_n F_n = \emptyset$ but $p_0 \in \bigcap_n \text{cl}_{\alpha X} f^{-1}[F_n]$.

Notice that in the next theorem we relax the condition that the image space be realcompact.

**Theorem D.** Let $X$ be dense in the first countable $T_1$-space $\alpha X$ and let $f: X \to Y$ continuously where $Y$ is completely regular. Then $f$ has a continuous extension $f^*: \alpha X \to Y$ if and only if for any two disjoint closed sets $F_1$ and $F_2$,
\[
\text{cl}_{\alpha X} f^{-1}[F_1] \cap \text{cl}_{\alpha X} f^{-1}[F_2] = \emptyset.
\]

**Proof. Sufficiency.** Suppose $f: X \to Y$ has no extension. Then arguing as before, $f$ does have an extension $f^*: \alpha X \to \beta Y$ and there must be a $p_0 \in \alpha X - X$ such that $f^*(p_0) \in \beta Y - Y$. (Here $\beta Y$ denotes the maximal Stone-Čech compactification of $Y$ which exists if and only if the space $Y$ is completely regular. See [3].)

Let $\{p_n\}$ be a sequence in $X$ with $p_n \to p_0$. We can then select a subsequence $\{x_n\} \subset \{p_n\}$ such that the points in $\{f^*(x_n)\}$ are distinct and $f^*(x_n) \to f^*(p_0)$. Consider the closed sets
\[
F_1 = \{f^*(x_1), f^*(x_3), \ldots \}
\]

and
Then \( F_1 \cap F_2 = \emptyset \), but \( \text{cl}_{aX} f^{-1}[F_1] \cap \text{cl}_{aX} f^{-1}[F_2] \neq \emptyset \).

**Necessity.** Let \( f \) have an extension \( f^*: aX \to Y \) and suppose for two closed \( F_1, F_2, p_0 \in \text{cl}_{aX} f^{-1}[F_1] \cap \text{cl}_{aX} f^{-1}[F_2] \). We then have that

\[
\begin{align*}
    f^*(p_0) \in f^*[\text{cl}_{aX} f^{-1}[F_1]] \cap f^*[\text{cl}_{aX} f^{-1}[F_2]] \\
    \subseteq \text{cl}_Y f^*[f^{-1}[F_1]] \cap \text{cl}_Y f^*[f^{-1}[F_2]] \\
    = \text{cl}_Y F_1 \cap \text{cl}_Y F_2 = F_1 \cap F_2,
\end{align*}
\]

so their intersection is nonempty.

**Added in proof.** After this paper was submitted for publication, the authors have discovered that Theorem A has also been proved by R. Engelking [4]. On the other hand, our method can be used to prove the following generalization of Theorems A and C (the terminology is that of [5] and [6]).

**Theorems A’ & C’.** Let \( E \) be a Hausdorff space having an \( E \)-completely regular compactification \( E^* \) such that every point of \( E^* - E \) has a local base of cardinality \( m \). Let \( Y \) be \( E \)-compact, let \( X \) be a dense subspace of a \( T_1 \)-space \( aX \) and let \( f \) be a continuous function with \( f: X \to Y \). Then the following are equivalent:

(a) \( f \) admits a continuous extension \( f^*: aX \to Y \);

(b) for every class \( \mathcal{K} \) of closed subsets of \( Y \) with \( \text{card} \ \mathcal{K} \leq m \) and \( \bigcap \mathcal{K} = \emptyset \) we have \( \bigcap \{ \text{cl}_{aX} f^{-1}[F]: F \in \mathcal{K} \} = \emptyset \);

(c) for every two disjoint closed subsets \( F_1 \) and \( F_2 \) of \( Y \) we have \( \text{cl}_{aX} f^{-1}[F_1] \cap \text{cl}_{aX} f^{-1}[F_2] = \emptyset \) and for every submultiplicative class \( \mathcal{K} \) of closed subsets of \( Y \) with \( \text{card} \ \mathcal{K} \leq m \) and \( \bigcap \mathcal{K} = \emptyset \) we have \( \bigcap \{ \text{cl}_{aX} f^{-1}[F]: F \in \mathcal{K} \} = \emptyset \).

(A class \( \mathcal{K} \) of sets is called submultiplicative provided that for every \( F_1, F_2 \in \mathcal{K} \) there is an \( F_3 \in \mathcal{K} \) with \( F_3 \subseteq F_1 \cap F_2 \).)

Note that the nontrivial part of Theorem A’ and C’ [that (c) implies (a)] can be easily derived from the statement at the end of §2 in [6]. Indeed, by the first part of (c) (and by the quoted Taimanov theorem) we infer that \( f \) admits a continuous extension \( f^*: aX \to \beta_{E^*} Y \).

Then, using the second part of (c) and the quoted statement of [6], we obtain that actually \( f^*[aX] \subset Y \).

Finally, let us call a subset \( F \) of a space \( Y \) \( E \)-closed provided that there exists a continuous function \( f: Y \to E^n \), where \( n \) is finite, and a closed subset \( A \) of \( E^n \) with \( F = f^{-1}[A] \). Theorems A’ & C’ remain true if the phrase “closed subsets of \( Y \)” is replaced by “\( E \)-closed subsets of \( Y \)”.
Bibliography


University of Rhode Island and
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