A SELECTION THEOREM

E. MICHAEL

1. Introduction. The following theorem was proved in [1, Footnote 7]. (A function $\phi$ from $X$ to the collection $2^B$ of nonempty closed subsets of $B$ is called lower semicontinuous (= l.s.c.) if $\{x \in X : \phi(x) \cap V \neq \emptyset\}$ is open in $X$ whenever $V$ is open in $B$, while $\Gamma_B A$ denotes the closed convex hull of $A$ in $B$.)

Theorem 1.1 [1]. If $X$ is paracompact, if $B$ is a Banach space, and if $\phi : X \to 2^B$ is l.s.c., then there is a continuous $f : X \to Y$ such that $f(x) \in \Gamma_B \phi(x)$ for every $x \in X$.

As was pointed out in [1, p. 364], Theorem 1.1 remains true if $B$ is any complete, metrizable locally convex space, but it is generally false if $B$ is not metrizable. We can, however, prove the following generalization of Theorem 1.1.

Theorem 1.2. Let $X$ be paracompact, and $M$ a metrizable subset of a complete locally convex space $E$. Let $\phi : X \to 2^M$ be l.s.c. and such that, for some metric on $M$, every $\phi(x)$ is complete. Then there exists a continuous $f : X \to E$ such that $f(x) \in \Gamma_E \phi(x)$ for every $x \in X$.

Theorem 1.2 was proved in [3] under the stronger assumption that $X$ is metrizable. While that was sufficient for the applications in [3], and probably for most other applications, it did not generalize Theorem 1.1, and was therefore never entirely satisfying. In this
paper, some machinery is created in §2 which enables us to prove Theorem 1.2 in full generality.

2. Two lemmas. Let $M$ be a metric space with metric $\rho$, and $L$ the linear space of real-valued Lipschitz functions on $X$. As was shown in [4], there is a Banach space $B$ containing $M$ isometrically, and a linear map $f \mapsto f^*$ from $L$ to the dual space of $B$, such that $f^*|M=f$ for all $f \in L$.

For each $f \in L$, let $s(f) = \{ x \in M : f(x) \neq 0 \}$ for each $y \in B$, let

$$
S(y) = \{ U \subseteq M : U \text{ open}, f^*(y) = 0 \text{ whenever } f \in L \text{ and } s(f) \subseteq U \},
$$

and let $\sigma(y) = M - \bigcup S(y)$. Clearly $\sigma(y)$ is closed.

The following lemmas are not the sharpest results possible, but they suffice for our purposes.

**Lemma 2.1.** Suppose that $K \subseteq M$ is compact, and that $y \in \Gamma_B K$. Then

(a) $\sigma(y) \subseteq K$.

(b) If $f \in L$, and $\sigma(y) \cap s(f) = \emptyset$, then $f^*(y) = 0$.

(c) $\sigma(y) \neq \emptyset$.

**Proof.** (a) Let $U = M - K$. Then $f^*(y) = 0$ whenever $s(f) \subseteq U$, so $U \subseteq S(y)$ and hence $\sigma(y) \subseteq K$.

(b) Since $f = f^+ - f^-$, where $f^+ \geq 0$ and $f^- \geq 0$ and both are in $L$, we need only prove (b) for $f^0$.

Let $A = s(f) \cap K$. Then $A$ is compact and disjoint from $\sigma(y)$, so it can be covered by open $V_i \subseteq X$ $(i = 1, \ldots, n)$ such that $\overline{V_i} \subseteq U_i$ for some $U_i \subseteq S(y)$. Let $g_i(x) = \rho(x, X - V_i)$ for all $x \in M$. Then $g_i \in L$ and $s(g_i) \subseteq U_i$, so $g_i^*(y) = 0$. Let $g = \sum_{i=1}^{n} g_i$. Then $g^*(y) = 0$. Now $g(x) > 0$ when $x \in A$, so there is an $\alpha > 0$ such that $0 \leq f(x) \leq \alpha g(x)$ for all $x \in A$, and therefore for all $x \in K$ since $f(x) = 0$ if $x \in K - A$. Hence $0 \leq f^*(y) \leq \alpha g^*(y) = 0$.

(c) Let $h(x) = 1$ for all $x \in M$. Then $h \in L$ and $h^*(y) = 1$, so $\sigma(y) \cap s(h) \neq \emptyset$ by (b). Hence $\sigma(y) \neq \emptyset$, and that completes the proof.

Now let $H = \bigcup \{ \Gamma_B K : K \subseteq M, K \text{ compact} \}$, let $\mathcal{K}(M)$ be the set of compact elements of $2^M$, and let $\sigma : H \to \mathcal{K}(M)$ be the map $y \mapsto \sigma(y)$.

**Lemma 2.2.** The map $\sigma : H \to \mathcal{K}(M)$ is l.s.c.

**Proof.** Let $V \subseteq M$ be open, and suppose that $y_0 \in H$ and $\sigma(y_0) \cap V \neq \emptyset$. Then there is an $f_0 \in L$ such that $s(f_0) \subseteq V$ and $f_0^*(y_0) \neq 0$. Let $U = \{ y \in H : f_0^*(y) \neq 0 \}$. Then $U$ is a neighborhood $y_0$ in $H$, and if $y \in U$, then $\sigma(y) \cap s(f_0) \neq \emptyset$ by Lemma 2.1 (b), so $\sigma(y) \cap V \neq \emptyset$. Hence $\{ y \in H : \sigma(y) \cap V \neq \emptyset \}$ is open in $H$, so $\sigma$ is l.s.c.

3. Proof of Theorem 1.2. First, apply [2, Theorem 1.1] to pick a
l.s.c. map $\psi: X \to \mathcal{P}(M)$ such that, for all $x \in X$, we have $\psi(x) \subseteq \emptyset(x)$ and $\psi(x)$ is compact.

Let $B \supset H \supset M$ be as in §2, and apply Theorem 1.1 to pick a continuous $g: X \to B$ such that $g(x) \in \Gamma_B \psi(x) \ for \ all \ x \in X$. Then $g(x) \in H$ for all $x \in X$.

Let $\sigma: H \to \mathcal{K}(M)$ be as in §2. Apply [3, Theorem 1.2] (that is, our Theorem 1.2 with metrizable domain) to the set-valued function $\sigma$, which is l.s.c. by Lemma 2.2, to pick a continuous $h: H \to E$ such that $h(y) \in \Gamma_E \sigma(y)$ for all $y \in H$.

Define $f: X \to E$ by $f = goh$. If $x \in X$, then $g(x) \in \Gamma_B \psi(x)$, so $\sigma(g(x)) \subseteq \psi(x) \subseteq \emptyset(x)$ by Lemma 2.1 (a), and hence

$$f(x) = h(g(x)) \in \Gamma_E \sigma(g(x)) \subseteq \Gamma_E \emptyset(x).$$

That completes the proof.

References


University of Washington