A SELECTION THEOREM

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1. Introduction. The following theorem was proved in [1, Footnote 7]. (A function $\phi$ from $X$ to the collection $2^B$ of nonempty closed subsets of $B$ is called lower semicontinuous (= l.s.c.) if $\{x \in X : \phi(x) \cap V \neq \emptyset\}$ is open in $X$ whenever $V$ is open in $B$, while $\Gamma_B A$ denotes the closed convex hull of $A$ in $B$.)

**Theorem 1.1** [1]. If $X$ is paracompact, if $B$ is a Banach space, and if $\phi : X \rightarrow 2^B$ is l.s.c., then there is a continuous $f : X \rightarrow Y$ such that $f(x) \in \Gamma_B \phi(x)$ for every $x \in X$.

As was pointed out in [1, p. 364], Theorem 1.1 remains true if $B$ is any complete, metrizable locally convex space, but it is generally false if $B$ is not metrizable. We can, however, prove the following generalization of Theorem 1.1.

**Theorem 1.2.** Let $X$ be paracompact, and $M$ a metrizable subset of a complete locally convex space $E$. Let $\phi : X \rightarrow 2^M$ be l.s.c. and such that, for some metric on $M$, every $\phi(x)$ is complete. Then there exists a continuous $f : X \rightarrow E$ such that $f(x) \in \Gamma_B \phi(x)$ for every $x \in X$.

Theorem 1.2 was proved in [3] under the stronger assumption that $X$ is metrizable. While that was sufficient for the applications in [3], and probably for most other applications, it did not generalize Theorem 1.1, and was therefore never entirely satisfying. In this

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2 It suffices if $\Gamma_B K$ is compact for every compact $K \subseteq M$. 

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paper, some machinery is created in §2 which enables us to prove Theorem 1.2 in full generality.

2. Two lemmas. Let $M$ be a metric space with metric $\rho$, and $L$ the linear space of real-valued Lipschitz functions on $X$. As was shown in [4], there is a Banach space $B$ containing $M$ isometrically, and a linear map $f \mapsto f^*$ from $L$ to the dual space of $B$, such that $f^*|M = f$ for all $f \in L$.

For each $f \in L$, let $s(f) = \{ x \in M : f(x) \neq 0 \}$. For each $y \in B$, let $s(y) = \{ U \subset M : U \text{ open}, f^*(y) = 0 \text{ whenever } f \in L \text{ and } s(f) \subset U \}$, and let $\sigma(y) = M - \bigcup s(y)$. Clearly $\sigma(y)$ is closed.

The following lemmas are not the sharpest results possible, but they suffice for our purposes.

**Lemma 2.1.** Suppose that $K \subset M$ is compact, and that $y \in \Gamma_B K$. Then

(a) $\sigma(y) \subset K$.

(b) If $f \in L$, and $\sigma(y) \cap s(f) = \emptyset$, then $f^*(y) = 0$.

(c) $\sigma(y) \neq \emptyset$.

**Proof.** (a) Let $U = M - K$. Then $f^*(y) = 0$ whenever $s(f) \subset U$, so $U \subset \sigma(y)$ and hence $\sigma(y) \subset K$.

(b) Since $f = f^+ - f^-$, where $f^+ \geq 0$ and $f^- \geq 0$ and both are in $L$, we need only prove (b) for $f^+ \geq 0$.

Let $A = s(f) \cap K$. Then $A$ is compact and disjoint from $\sigma(y)$, so it can be covered by open $V_i \subset X$ $(i = 1, \ldots, n)$ such that $\overline{V_i} \subset U_i$ for some $U_i \subset s(y)$. Let $g_i(x) = \rho(x, X - V_i)$ for all $x \in M$. Then $g_i \in L$ and $s(g_i) \subset U_i$, so $g_i^*(y) = 0$. Let $g = \sum g_i$. Then $g^*(y) = 0$. Now $g(x) > 0$ when $x \in A$, so there is an $\alpha > 0$ such that $0 \leq f(x) \leq \alpha g(x)$ for all $x \in A$, and therefore for all $x \in K$ since $f(x) = 0$ if $x \in K - A$. Hence $0 \leq f^*(y) \leq \alpha g^*(y) = 0$.

(c) Let $h(x) = 1$ for all $x \in M$. Then $h \in L$ and $h^*(y) = 1$, so $\sigma(y) \cap s(h) \neq \emptyset$ by (b). Hence $\sigma(y) \neq \emptyset$, and that completes the proof.

Now let $H = \bigcup \{ \Gamma_B K : K \subset M, K \text{ compact} \}$, let $\mathcal{K}(M)$ be the set of compact elements of $2^M$, and let $\sigma : H \rightarrow \mathcal{K}(M)$ be the map $y \mapsto \sigma(y)$.

**Lemma 2.2.** The map $\sigma : H \rightarrow \mathcal{K}(M)$ is l.s.c.

**Proof.** Let $V \subset M$ be open, and suppose that $y_0 \in H$ and $\sigma(y_0) \cap V \neq \emptyset$. Then there is an $f_0 \in L$ such that $s(f_0) \subset V$ and $f_0^*(y_0) \neq 0$. Let $U = \{ y \in H : f_0^*(y) \neq 0 \}$. Then $U$ is a neighborhood $y_0$ in $H$, and if $y \in U$, then $\sigma(y) \cap s(f_0) \neq \emptyset$ by Lemma 2.1 (b), so $\sigma(y) \cap V \neq \emptyset$. Hence $\{ y \in H : \sigma(y) \cap V \neq \emptyset \}$ is open in $H$, so $\sigma$ is l.s.c.

3. Proof of Theorem 1.2. First, apply [2, Theorem 1.1] to pick a
l.s.c. map $\psi: X \to 2^M$ such that, for all $x \in X$, we have $\psi(x) \subseteq \emptyset(x)$ and $\psi(x)$ is compact.

Let $B \supset H \supset M$ be as in §2, and apply Theorem 1.1 to pick a continuous $g: X \to B$ such that $g(x) \in \Gamma_B \psi(x)$ for all $x \in X$. Then $g(x) \in H$ for all $x \in X$.

Let $\sigma: H \to \mathcal{K}(M)$ be as in §2. Apply [3, Theorem 1.2] (that is, our Theorem 1.2 with metrizable domain) to the set-valued function $\sigma$, which is l.s.c. by Lemma 2.2, to pick a continuous $h: H \to E$ such that $h(y) \in \Gamma_E \sigma(y)$ for all $y \in H$.

Define $f: X \to E$ by $f = gh$. If $x \in X$, then $g(x) \in \Gamma_B \psi(x)$, so $\sigma(g(x)) \subseteq \psi(x) \subseteq \emptyset(x)$ by Lemma 2.1 (a), and hence

$$f(x) = h(g(x)) \in \Gamma_E \sigma(g(x)) \subseteq \Gamma_E \emptyset(x).$$

That completes the proof.

References


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