On the Permanent and Maximal Characteristic Root of a Nonnegative Matrix

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One of the many theorems that M. Marcus and M. Newman prove in \[3\] is, when it is restricted to nonnegative matrices, that: if \( A \) is an \( n \times n \) positive semidefinite symmetric irreducible nonnegative matrix, then

\[
\lim_{m \to \infty} \left( \text{per}(A^m) \right)^{1/m} = r^n
\]

where \( r \) denotes the maximal (positive) characteristic root of \( A \). Here \( \text{per}(A^m) \) denotes the permanent of \( A^m \). We assume that the reader is familiar with the terminology and results of the classical Perron-Frobenius-Wielandt theory of nonnegative matrices, the requisite parts of which can be found in \[1, \text{Chapter XIII}\].

The purpose of this note is to prove the following extension of the result quoted above.

\textbf{Theorem.} Let \( A \) be an \( n \times n \) nonnegative irreducible matrix and suppose \( A \) has a nonzero permanent. Let \( r \) be the maximal characteristic root of \( A \). Then

\[
\lim_{m \to \infty} \left( \text{per}(A^m) \right)^{1/m} = r^n.
\]

We first prove two lemmas.

\textbf{Lemma 1.} Let \( B \) be an \( n \times n \) nonnegative matrix with maximal characteristic root \( \rho \). Then

\[
\text{per}(B) \leq \rho^n.
\]

\textbf{Proof.} First assume \( A \) is positive (or irreducible). Then \( \rho \) is positive and has an associated characteristic vector \( x = (x_1, \ldots, x_n) \), all of whose coordinates are positive. From \( Bx = \rho x \), it follows that

\[
\sum_{j=1}^{n} b_{ij}x_j = \rho x_i, \quad i = 1, \ldots, n
\]

and hence

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2 This is a slight variation of a proof originally shown the author by Morris Newman.
This may be written as

\[ \text{per}(B) \prod_{i=1}^{n} x_i + \beta = \rho^n \prod_{i=1}^{n} x_i \]

where \( \beta \geq 0 \) and the result follows. The complete lemma now follows by continuity.

**Lemma 2.** Let \( A \) be an \( n \times n \) nonnegative primitive matrix with maximal characteristic root \( r \). Then (1) holds.

**Proof.** If \( m \) is a positive integer, then using Lemma 1 and the fact that \( A^m \) is nonnegative, we may write

\[ \prod_{i=1}^{n} a_{ii}^{(m)} \leq \text{per}(A^m) \leq (r^m)^n \]

where \( a_{ij}^{(m)} \) denotes the \((i, j)\) entry of \( A^m \). Thus we have

\[ \prod_{i=1}^{n} (a_{ii}^{(m)})^{1/m} \leq \text{per}(A^m)^{1/m} \leq r^n. \]

But it is well known (see [2, p. 128] or [1, p. 81]) that for a primitive matrix \( A \),

\[ \lim_{m \to \infty} (a_{ij}^{(m)})^{1/m} = r. \]

Taking limits in (2), we obtain (1).

**Proof of Theorem.** Let \( h \) denote the index of imprimitivity of \( A \), that is, the number of characteristic roots of \( A \) of modulus \( r \). Then [1, p. 81–82] there is a permutation matrix \( P \) such that

\[ P^T A^h P = \text{diag}(A_1, A_2, \ldots, A_h) \]

where each \( A_i \), \( 1 \leq i \leq h \), is an \( n_i \times n_i \) primitive matrix with maximal characteristic root \( r^h \). Observe that if \( k \) is a positive integer, then

\[ (P^T A^h P)^k = P^T A^{hk} P = \text{diag}(A_1^k, A_2^k, \ldots, A_h^k). \]

Since the permanent of a matrix is invariant under permutations, we have

\[ \text{per}(A^{hk})^{1/hk} = [(\text{per}(A_1^k))^{1/k} \cdots (\text{per}(A_h^k))^{1/k}]^{1/h}. \]
Since each $A_i$ is a primitive matrix, it follows by Lemma 2, that

$$\lim_{k \to \infty} (\text{per}(A^{hk}))^{1/hk} = \left(\prod_{i=1}^{h} (r^h)^{n_i}\right)^{1/h} = r^n.$$  \hfill (3)

Now let $m$ be an arbitrary positive integer and write

$$m = hp_m + q_m; \quad p_m \geq 0, \quad 0 \leq q_m < h.$$  

Then

$$A^m = A^{hp_m}A^{qm}$$

where $A^{qm} = I$ if $q_m = 0$. Thus since all matrices are nonnegative

$$\text{per}(A^m) \geq \text{per}(A^{hp_m}) \text{ per}(A^{qm}).$$

From (3) it follows immediately that

$$\lim_{m \to \infty} \left(\text{per}(A^{hp_m})\right)^{1/hp_m} = r^n.$$  \hfill (4)

But then using (4) we obtain

$$\lim_{m \to \infty} \left(\frac{\ln(\text{per}(A^{hp_m})\cdot \text{ per}(A^{hp_m}))}{\ln(\text{per}(A^{hp_m}))^{1/hp_m}}\right) = \lim_{m \to \infty} \frac{q_m}{m} \ln(\text{per}(A^{hp_m}))^{1/hp_m} = 0.$$

Hence

$$\lim_{m \to \infty} \frac{(\text{per}(A^{hp_m}))^{1/hp_m}}{(\text{per}(A^{hp_m}))^{1/m}} = 1.$$  \hfill (5)

Now (4) and (5) together imply

$$\lim_{m \to \infty} \left(\text{per}(A^{hp_m})\right)^{1/m} = r^n.$$  \hfill (6)

Now by assumption $\text{per}(A) > 0$. If $\text{per}(A) \geq 1$, then

$$\text{per}(A^{qm}) \geq (\text{per}(A))^{qm} \geq 1, \quad \text{for all } m;$$

while if $\text{per}(A) < 1$, then

$$\text{per}(A^{qm}) \geq (\text{per}(A))^{qm} \geq \text{ per}(A)^{q_m}, \quad \text{for all } m.$$

If $r \geq 1$, then by Lemma 1

$$\text{per}(A^{qm}) \leq (r^n)^{q_m} \leq r^n(q-1), \quad \text{for all } m;$$

while if $r \leq 1$, then
per \((A^{qm}) \leq (r^n)^q m \leq 1\), for all \(m\).

Thus in any case there are positive constants \(c_1\) and \(c_2\) such that
\[
c_1 \leq \text{per } (A^{qm}) \leq c_2
\]
for all positive integers \(m\). From this it now follows that
\[
\lim_{m \to \infty} \left(\text{per } (A^{qm})\right)^{1/m} = 1.
\]

Passing to the limit in the inequality
\[
\left(\text{per } (A^{kp_m})\right)^{1/m} \left(\text{per } (A^{qm})\right)^{1/m} \leq \left(\text{per } (A^n)\right)^{1/m} \leq r^n,
\]
and using (6) and (7), we obtain the desired result.

The condition that \(A\) have a nonzero permanent cannot in general be omitted. The irreducible matrix
\[
A = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}
\]
has the property that \(\text{per } (A^k) = 0\) for \(k\) odd and \(\text{per } (A^k) > 0\) for \(k\) even. It is easy to verify that \(A\) has maximal characteristic root \(2^{1/2}\) and index of imprimitivity 2. It follows from (3) that if \(\lim_{m \to \infty} \left(\text{per } (A^n)^{1/m}\right)\) exists, it must be \(2^{3/2}\). Hence the limit does not even exist.

Finally we remark that our theorem does constitute an extension of the result of Marcus and Newman mentioned in the beginning. For, if \(A\) is a positive semidefinite symmetric irreducible nonnegative matrix, then the entries on the main diagonal of \(A\) must be positive. Otherwise \(A\) has a zero row and thus is not irreducible. Hence the permanent of such a matrix is positive.

References


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