A CLASS OF BANACH ALGEBRAS WITH A UNIQUE NORM TOPOLOGY

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The purpose of this paper is to produce a class of nonsemisimple Banach algebras with a unique norm topology. That is, a class of Banach algebras such that each member $B$ of the class has the property that any two Banach algebra norms on $B$ are equivalent (cf. [9, Chapter II]). (Throughout this paper, an algebra will denote any commutative algebra over the complex field $C$ which possesses an identity $e$.) To do this, we investigate an algebraic extension of a semisimple algebra $A$ which is similar to the Arens-Hoffman extension of a normed algebra (cf. [1]). We let $\alpha(x)$ be a monic polynomial in $A[x]$, the algebra of all polynomials in the indeterminate $x$ with coefficients in $A$, and denote by $(\alpha(x))$ the principal ideal in $A[x]$ generated by $\alpha(x)$. If $B = A[x]/(\alpha(x))$ is a Banach algebra with respect to some norm, then $A$ is a normed algebra with respect to the norm on $B$ restricted to $A$ and we ask whether or not $A$ is a closed subalgebra of $B$. If for any Banach algebra norm on $B$, $A$ is a closed subalgebra of $B$, then we show that $B$ has a unique norm topology.

The main result of this paper is that if $A$ is a regular (in the language of [9], completely regular), semisimple Banach algebra and if $\alpha(x)$ is any monic polynomial in $A[x]$, then $B = A[x]/(\alpha(x))$ has a unique norm topology, where $B$ is the Arens-Hoffman extension of $A$. (See below for a discussion of the Arens-Hoffman extension.) An example is given which shows that the condition of semisimplicity is essential. It is an open question whether or not the main result is true if $A$ is not a regular Banach algebra.

In the event that $A$ is a semisimple Banach algebra, $\alpha(x)$ is a monic polynomial in $A[x]$ and $B = A[x]/(\alpha(x))$ is semisimple, then by [9, Corollary 2.5.18], $B$ has a unique norm topology. Therefore our results have meaning only if $B$ is not semisimple. R. Arens and K. Hoffman have shown that if the discriminant $d$ of $\alpha(x)$ is not a zero divisor in $A$, then $B$ is semisimple [1, Theorem 4.3]. (See [1, p. 207] for the definition of $d$.) The converse of this latter result is also valid.

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and is proved by J. Lindberg [6, Corollary 9.3]. Therefore $B$ is not semisimple if and only if $d$ is either zero or a divisor of zero.

Unless otherwise stated, $A$ will denote a semisimple algebra, $\alpha(x)$ a monic polynomial in $A[x]$, and $B = A[x]/(\alpha(x))$. We will denote the coset $x + (\alpha(x))$ in $B$ by $x$ and the coset $a + (\alpha(x))$ by $a$, for each $a \in A$. Thus any element $b \in B$ can be uniquely expressed in the form $b = \sum_{i=0}^{n-1} a_i x^i$, where $n$ is the degree of $\alpha(x)$ and $a_0, a_1, \ldots, a_{n-1}$ are elements in $A$.

We begin by presenting a brief discussion of the Arens-Hoffman extension. If $A$ is a normed algebra (not necessarily semisimple), and $\alpha(x)$ is a monic polynomial in $A[x]$, Arens and Hoffman have shown in [1] that $B = A[x]/(\alpha(x))$ possesses a family of norms each of which makes $B$ into a normed algebra with the property that the natural embedding of $A$ into $B$ is an isometry as well as an isomorphism of $A$ onto a closed subalgebra of $B$. Furthermore, $B$ is a Banach algebra whenever $A$ is a Banach algebra. If $\alpha(x) = x^n + \sum_{i=0}^{n-1} a_i x^i$, then the family of norms is given by

$$
\left\| \sum_{i=0}^{n-1} a_i x^i \right\| = \sum_{i=0}^{n-1} \| a_i \| t^i,
$$

where $\| \cdot \|$ is the norm on $A$ and $t$ is any positive number such that $\sum_{i=0}^{n-1} \| a_i \| t^i \leq t^n$. Since there always exists a $t$ with the above property, we shall assume $t = 1$. We will refer to $B$ as an Arens-Hoffman extension of $A$ and call the norm

$$
\left\| \sum_{i=0}^{n-1} a_i x^i \right\| = \sum_{i=0}^{n-1} \| a_i \| t^i,
$$

for any appropriate $t$, an Arens-Hoffman norm.

We denote by $\Phi_A$ the carrier space of the normed algebra $A$ and by $\hat{A}$ the Gelfand representation of $A$ (cf. [9, Chapter III]). If $h \in \Phi_A$, we view $x$ as an indeterminate over $C$ as well as an indeterminate over $A$ and let $\alpha_h(x)$ denote the polynomial $\sum_{i=0}^{n-1} \alpha_i(h) x^i$ in $C[x]$. Arens and Hoffman have shown that if $A$ is a normed algebra and if $\alpha(x)$ is a monic polynomial in $A[x]$, then $\Phi_B(B = A[x]/(\alpha(x)))$ is identifiable with the set \{$(h, \lambda) \in \Phi_A \times C: \alpha_h(\lambda) = 0$\} together with the relative topology of $\Phi_A \times C$ [1, Theorem 4.2]. The Arens-Hoffman extension has been studied by G. A. Heuer in [4], J. A. Lindberg in [6], [7], and [8], and by Heuer and Lindberg in [5].

We now return to our assumption that $A$ is a semisimple algebra. Our first result shows that if $B$ is a Banach algebra, then there exists a norm on $A$ with respect to which $A$ is a Banach algebra.
Lemma 1. Assume $B$ is a Banach algebra with norm $\| \cdot \|$. Then there is a norm $\| \cdot \|'$ on $A$ with respect to which $A$ is a Banach algebra. Also, $\| \cdot \|'$ is majorized by the norm in $B$ restricted to $A$.

Proof. Let $R$ be the radical of $B$. Since $\hat{B}$ is isomorphic to $B/R$, $\hat{B}$ is a Banach algebra with respect to the quotient norm $\| \hat{b} \|_q = \inf_{r \in R} \| b + r \|$. We will show that $\hat{A}$ is a $\| \cdot \|_q$-closed subalgebra of $\hat{B}$. Assume $\{ \hat{a}_m \}$ is a Cauchy sequence in $\hat{A}$. Then there is an element $b \in B$ such that $\| \hat{a}_m - \hat{b} \|_q \to 0$ as $m \to \infty$. Since $\| \cdot \|_q$ majorizes the sup norm on $B$, $\| a_m - b \|_\infty \to 0$ as $m \to \infty$ and thus $\hat{b}$ is constant on each fiber of $\Phi_B$. For each $h \in \Phi_A$, denote the roots of $\alpha_h(x) = 0$ by $\lambda_1(h), \lambda_2(h), \ldots, \lambda_n(h)$, where each distinct root is repeated according to its multiplicity and $n$ is the degree of $\alpha(x)$. Also, let $s_1(h), s_2(h), \ldots, s_n(h)$ be the elementary symmetric functions of $\lambda_1(h), \lambda_2(h), \ldots, \lambda_n(h)$.

Since

$$\alpha_h(x) = \prod_{i=1}^n (x - \lambda_i(h)) = x^n + \sum_{j=1}^n (-1)^j s_j(h) x^{n-j}$$

for each $h \in \Phi_A$,

$$\hat{\alpha}(x) = x^n + \sum_{i=0}^{n-1} \hat{\alpha}_i x^i = x^n + \sum_{j=1}^n (-1)^j s_j x^{n-j}.$$

But this means each $s_j$ is an element of $\hat{A}$ since $\hat{\alpha}$ is a polynomial over $\hat{A}$.

Now define a function $f$ on $\Phi_A$ by $f(h) = \sum_{j=1}^n \hat{b}(h, \lambda_j(h))$ for each $h \in \Phi_A$. Clearly, $f \in C(\Phi_A)$. We will show that $f \in \hat{A}$. We view $f$ as a function on $\Phi_B$ by writing $f(h, \lambda_i(h)) = f(h)$ for any $(h, \lambda_i(h)) \in \Phi_B$. Thus $f$ is a symmetric function in the $\lambda_i(h)$'s and can be expressed as a polynomial in the $s_j$'s with coefficients in $\hat{A}$. Since each $s_j \in \hat{A}$, $f \in \hat{A}$. (This result is a special case of [8, Lemma 1.2].)

We have already shown that $\hat{b}$ is constant on the fibers of $\Phi_B$ and thus $f(h) = n \hat{b}(h, \lambda_j(h))$ for any $j = 1, 2, \ldots, n$. Therefore $\hat{b} \in \hat{A}$. We have thus shown that $\hat{A}$ is a closed subalgebra of $\hat{B}$.

Now define $\| a \|' = \| a \|_q$ for all $a \in A$. Since $A$ is isomorphic to $\hat{A}$, $\| \cdot \|'$ defines a norm on $A$ with respect to which $A$ is a Banach algebra. Furthermore, $\| a \|' = \inf_{r \in R} \| a + r \| \leq \| a \|$ and therefore the proof is complete.

We note that by [9, Corollary 2.5.18], $\| \cdot \|'$ is equivalent to any other Banach algebra norm on $A$.

Theorem 2. If $A$ has the property that for any Banach algebra norm on $B$, $A$ is a closed subalgebra of $B$, then $B$ has a unique norm topology.
Proof. Assume \( || \cdot || \) is a Banach algebra norm on \( B \). By Lemma 1, 
\( A \) is a Banach algebra with norm \( || \cdot ||' \) and for each \( a \in A, ||a||' \leq ||a|| \). 
Since \( A \) is a closed subalgebra of \( B \), by [9, Corollary 2.5.18], there exists a constant \( K \) such that \( ||a||' \leq ||a|| \leq K||a||' \) for each \( a \in A \). Let 
\( || \cdot ||_1 \) denote the Arens-Hoffman norm on \( B \) which extends \( || \cdot ||' \) on \( A \), i.e. \( \sum_{i=0}^{n-1} a_i \alpha^i || \cdot ||_1 = \sum_{i=0}^{n-1} ||a_i||' \). To show \( B \) has a unique norm topology, we show that \( || \cdot || \) is equivalent to \( || \cdot ||_1 \). If \( M = \max_{0 \leq i \leq n-1} ||\alpha^i|| \), then for any \( a_0, a_1, \ldots, a_{n-1} \) in \( A \),
\[
\sum_{i=0}^{n-1} a_i \alpha^i \leq M \sum_{i=0}^{n-1} ||a_i|| \leq MK \sum_{i=0}^{n-1} ||a_i||' = MK \sum_{i=0}^{n-1} a_i \alpha^i || \cdot ||_1,
\]
and, by the inverse mapping theorem, the two norms are equivalent. Therefore any norm on \( B \) is equivalent to the Arens-Hoffman norm and the proof is complete.

We now prove two conditions on \( A \) which are sufficient for \( A \) to be a closed subalgebra of \( B \). In the following theorem, we view \( A \) as a function algebra. This involves no loss of generality since \( A \) is assumed to be a semisimple algebra. Thus if \( B \) is a Banach algebra, by Lemma 1, \( A \) is also a Banach algebra and therefore \( A \) is a semisimple Banach algebra.

Theorem 3. Let \( X \) be a compact Hausdorff space and let \( A \) be a regular subalgebra of \( C(X) \) which contains the constant functions. If \( B \) is a Banach algebra with norm \( || \cdot || \), then \( A \) is a closed subalgebra of \( B \).

Proof. By Lemma 1, there exists a norm \( || \cdot ||' \) on \( A \) with respect to which \( A \) is a Banach algebra and \( ||a||' \leq ||a|| \) for each \( a \in A \). Since the identity mapping is an isomorphism of \( A \) into \( B \), by [2, Theorem 3.7 and Corollary 3.9], there is a finite set \( F = \{h_1, h_2, \ldots, h_k\} \) in \( \Phi_A \) such that for any neighborhood \( V \) in \( \Phi_A \) of \( F \), there is a constant \( M_V \) such that \( ||a|| \leq M_V ||a||' \) for all \( a \in K(V) \), where \( K(V) = \{c \in A: c = 0 \text{ on } V\} \) is the kernel of \( V \). For any such \( V \), let \( u_V \in A \) be such that \( u_V = 1 \) on \( V \).

To show \( A \) is a closed subalgebra of \( B \), let \( \{a_m\} \) be a Cauchy sequence in \( A \) with respect to \( || \cdot || \). Thus there exists an element \( b \in B \) such that \( ||a_m - b|| \to 0 \) as \( m \to \infty \). But \( \{a_m\} \) is also a Cauchy sequence with respect to \( || \cdot ||' \) and hence there exists an element \( c \in A \) such that \( ||a_m - c||' \to 0 \) as \( m \to \infty \). For any neighborhood \( V \) in \( \Phi_A \) of \( F, a_m - a_m u_V \in K(V) \) for each \( m = 1, 2, \ldots \), and thus \( ||a_m - a_m u_V|| \leq M_V ||a_m - a_m u_V||' \). Therefore \( ||a_m - a_m u_V - (c - cu_V)|| \to 0 \) as \( m \to \infty \) and hence \( c - cu_V = b - bu_V \). Let \( b_0, b_1, \ldots, b_{n-1} \) be in \( A \) such that \( b = \sum_{i=0}^{n-1} b_i \alpha^i \) where \( n \) is the degree of \( \alpha(x) \). Then \( c - cu_V = b_0 - b_0 u_V \).
and for \(i = 1, 2, \ldots, n-1\), \(b_i = b_i u_v\) which means \(b_i = 0\) on \(\{ h \in \Phi_A: \hat{a}_v(h) \neq 1 \}\).

To show \(b_i = 0\) for \(i = 1, 2, \ldots, n-1\), we first show \(b_i = 0\) on \(\Phi_A - F\). If \(h_0 \in \Phi_A - F\), let \(V\) be a closed neighborhood in \(\Phi_A\) of \(F\) such that \(h_0 \notin V\) and let \(u_v \in A\) such that \(\hat{a}_v = 1\) on \(V\) and \(\hat{a}_v(h_0) = 0\). These choices are possible since \(\Phi_A\) is compact Hausdorff and \(A\) is regular. Therefore \(b_i(h_0) = \hat{b}_i(h_0) \hat{a}_v(h_0) = 0\) for \(i = 1, 2, \ldots, n-1\), and since \(h_0\) was arbitrary, \(b_i = 0\) on \(\Phi_A - F\). We next show that each \(b_i = 0\) on \(F\).

Since each \(\hat{b}_i\) is continuous on \(\Phi_A\), each point \(h_j \in F\) such that \(\hat{a}_j(h_j) \neq 0\) for some \(j = 1, 2, \ldots, n-1\) is an isolated point of \(\Phi_A - F\). By relabeling these points if necessary, we may and do assume that there exists an integer \(k' \leq k\) such that for each \(j = 1, 2, \ldots, k'\), there is an \(i\) such that \(1 \leq i \leq n-1\) and \(\hat{b}_i(h_j) \neq 0\). Thus for each \(j = 1, 2, \ldots, k'\), there exists an idempotent \(u_j\) in \(A\) such that \(\hat{a}_j(h_j) = 1\) and \(\hat{a}_j = 0\) on \(\Phi_A - \{h_j\}\). For any such \(j\), consider the ideal \((u_j)\) in \(A\) generated by \(u_j\). This ideal is one-dimensional and thus is closed with respect to both norms in \(A\). Since \(a_m u_j \in (u_j)\) for each \(m\), \(b u_j \in (u_j) \subset A\). But \(b u_j = \sum_{i=0}^{n-1} b_i u_i t^i\) and thus for each \(i = 1, 2, \ldots, n-1\), \(b_i u_j = 0\).

Therefore each \(\hat{b}_i(h_j) = 0\) and this means, since \(j\) was arbitrary, that each \(\hat{b}_i\) is identically zero on \(F\). Since \(A\) is semisimple, each \(b_i = 0\) and therefore \(b = b_0 \in A\). Thus \(A\) is a closed subalgebra of \(B\) and the proof is complete.

The techniques used in the above proof to show that each \(\hat{b}_i\) is zero on \(F\) are used to prove the following theorem.

**Theorem 4.** Assume \(B\) is a Banach algebra with norm \(\| \cdot \|\). If the isolated points of \(\Phi_A\) are hull-kernel dense in \(\Phi_A\), then \(A\) is a closed subalgebra of \(B\).

**Proof.** Let \(I = \{ h_\gamma: \gamma \in \Gamma \}\) be the isolated points in \(\Phi_A\). For each \(\gamma \in \Gamma\), let \(e_\gamma\) be the idempotent in \(A\) such that \(\hat{e}_\gamma(h_\gamma) = 1\) and \(\hat{e}_\gamma = 0\) on \(\Phi_A - \{h_\gamma\}\). Let \(\{a_m\}\) be a Cauchy sequence in \(A\) and assume for \(b = \sum_{i=0}^{n-1} b_i t^i \in B\) that \(\|a_m - b\| \to 0\) as \(m \to \infty\). For each \(\gamma\), the ideal \((e_\gamma)\) in \(A\) generated by \(e_\gamma\) is one dimensional and hence is closed in \(A\) with respect to the norm in \(B\) restricted to \(A\). Since \(a_m e_\gamma \in (e_\gamma)\) for all \(m\), \(b e_\gamma \in (e_\gamma) \subset A\) and thus \(b_i e_\gamma = 0\) for \(i = 1, 2, \ldots, n-1\). But this means that \(\hat{b}_i(h_\gamma) = 0\) and, since \(\gamma\) was arbitrary, \(b_i \in K(I)\) for \(i = 1, 2, \ldots, n-1\). Therefore \(\hat{b}_i = 0\) on \(HK(I)\), the hull of the kernel of \(I\), which by assumption is \(\Phi_A\). Finally, since \(A\) is semisimple, \(b_i = 0\) and hence \(b = b_0 \in A\). This completes the proof.

We conclude with an example showing that the semisimplicity of \(A\) is essential in Theorem 3.

**Example.** Let \(A_1\) be any commutative Banach algebra without
identity having the property that there exists a dense subalgebra $A_0$ of $A_1$ such that $A_1 = A_0 + R$, where the radical of $A_1$ is $R$, and $R$ is a principal ideal generated by an element $r$ such that $r^2 = ra = 0$ for any $a \in A_0$. (See [2] or [3] for specific examples.) Thus $A_0$ is a Banach algebra with respect to the quotient norm $\|a\|_Q = \inf_{\gamma \in c} \|a + \gamma r\|$. Adjoin an identity $e$ to $A_1$ and obtain $A = A_1 \oplus \{\lambda e\}$. Let $B = A[x]/(x^2) = A \oplus A_1$ and define a norm on $B$ as follows.

$$\|a + \gamma r + \lambda e + (b + \mu r + ve)x\|_1 = \|a + \mu r\| + |\gamma| + |\lambda| + \|b\| + |v|$$

where $a, b \in A_0$. It is straightforward to verify that $B$ is a Banach algebra with norm $\| \cdot \|_1$. If we restrict $\| \cdot \|_1$ to $A$, we obtain $\|a + \gamma r + \lambda e\|_1 = \|a\| + |\gamma| + |\lambda|$. Since $A_0$ is not a closed subalgebra of $A_1$, there exists a Cauchy sequence $\{a_m\}$ in $A_0$ such that for some $a + \mu r$ in $A_1$ with $\mu \neq 0$, $\|a_m - a - \mu r\| \to 0$ as $m \to \infty$. Thus $\{a_m - a\}$ is a Cauchy sequence in $A$ which converges to $\mu r$ with respect to $\| \cdot \|_1$. Therefore $A$ is not a closed subalgebra of $B$ with respect to $\| \cdot \|_1$.

**Bibliography**


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