Let $S$ denote the classical family of schlicht functions $f$ on the unit disk $E$ which have the Taylor expansion $f(z) = z + \sum_{j=2}^{\infty} A_j z^j$. Recently Ozawa [5] used the Grunsky inequalities and an inequality of Jenkins to show that if $A_2 \geq 0$ then $\Re A_6 \leq 6$ with equality occurring only for the Koebe slit function $z/(1-z)^2$. In this note we shall show that if $A_j$ is real for $j \leq p$ then $\Re A_n \leq n$ for $n \leq 2p + 1$ with equality occurring only for one of the Koebe slit functions $z/(1 \pm z)^2$. This will be established by using a continuity argument to deduce from Jenkins' General Coefficient Theorem [1] that the extremal functions for this coefficient problem have real coefficients.

Let $S_p = \{f \in S: A_j \text{ is real for } j \leq p\}$, $S_\infty = \{f \in S: A_j \text{ is real for every } j\}$. Set $V_{p,n} = \{(\Re A_1, \cdots, \Re A_n): f \in S_p\}$. Let $H_{n,\varepsilon}$ ($\varepsilon = \pm 1$) denote the metric space of symmetric pairs $(\Omega, g)$ defined as follows. First $\Omega = P(w)dw^2$ is a quadratic differential on the Riemann sphere $R$ of the canonical form

\begin{align*}
(1) & \quad P(\bar{w}) = P(w), \\
(2) & \quad P(w) = \alpha K \left[ \prod_{j=1}^{s} \frac{(w - a_j)/w^{s+1}}{w^{r+1}} \right] dw^2,
\end{align*}

where $\alpha = \pm 1$, $K > 0$, $2 \leq s \leq n$, $0 \leq r \leq s - 2$ (we adopt the convention that $\prod_{j=m}^{k} u_j = 1$ if $m < k$) and where $\alpha = \varepsilon$ if $s = n$. Second $g \in S_\infty$. (Notice that $g \in S_\infty$ if and only if $g \in S$ and $g(z) = f(z)$.) Third $g$ and $\Omega$ are associated [3]. (In other words $g(E)$ is an admissible domain with respect to $\Omega$ in the sense of Jenkins.) Finally the metric $d$ on $H_{n,1} \cup H_{n,-1}$ is defined by the equation

$$d((\Omega_1, g_1), (\Omega_2, g_2)) = \sup\{ \left| P_1(w) - P_2(w) \right| : |w| = 1 \} + \sup\{ \left| g_1(z) - g_2(z) \right| : |z| = 1/2 \}. $$

The pairs of $H_{n,1}$, $H_{n,-1}$ will be denoted by $(\Omega_*, g_*)$, $(\Omega_{**}, g_{**})$ instead of $(\Omega, g)$. The coefficients of $g_*$, $g_{**}$ will be denoted by $B_j$, $C_j$ instead of $A_j$.

**Theorem 1.** If $p \geq 1$ and $n \leq 2p$ then $V_{p,n}$ is homeomorphic to a closed ball in $R^{n-1}$, the real Euclidean space of $n - 1$ dimensions. Every point of $\partial V_{p,n}$, the topological boundary of $V_{p,n}$, is taken by a unique slit function in $S_\infty$. Every point of $\text{int } V_{p,n}$, the interior of $V_{p,n}$, is taken

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by infinitely many bounded schlicht functions in $S_p$. In addition

\[ V_{p,n} = \{(B_1, \ldots, B_n): (\Omega_*, g_*) \in H_{n+1,1}\} \]

\[ = \{(C_1, \ldots, C_n): (\Omega_**, g**) \in H_{n+1,-1}\}. \]

**Proof.** When $p=1$ the facts in Theorem 1 are either trivial or else follow immediately from well-known facts. Suppose Theorem 1 is true for $p \leq q-1$. Then if $p=q$ and $n \leq 2q-2$ the truth of Theorem 1 follows since $S_q \subset S_{q-1}$ and $V_{q,n} = V_{q-1,n}$. Therefore if $f \in S_q$ then $(\Omega_*, g_*)$, $(\Omega_**, g**) \in H_{2q-1,1}, H_{2q-1,-1}$ such that

\[ (B_1, \ldots, B_{2q-2}) = (C_1, \ldots, C_{2q-2}). \]

If $\Omega_*$ or $\Omega_**$ has a pole at the origin of order less than $2q$ then by the induction hypothesis $g_*=f=g**$, $(\Re A_1, \ldots, \Re A_{2q-2}) \in \partial V_{q,2q-2}$ and hence $(\Re A_1, \ldots, \Re A_{2q-1}) \in \partial V_{q,2q-1}$. Therefore we may assume that the poles are of order $2q$ and $(\Re A_1, \ldots, \Re A_{2q-2}) \in \text{int } V_{q,2q-2}$ is taken by a bounded schlicht function $b \in S_p$.

Next we apply the General Coefficient Theorem in its current form [1] to $R$ the Riemann sphere and $\Omega, g(E), f \circ g^{-1}$, where $g=g_*$ or $g(**$, $\Omega=\Omega_*$ or $\Omega_**$. Admissible homotopies into the identity are abundant, $g(E)$ is admissible by its definition, and $g \circ f^{-1}$ is admissible as well. In fact if $w$ is a suitable parameter representing the origin as the point at infinity then by (1), (2), (3) we obtain the expansions at infinity

\[ P(1/w)(-1/w^2)^2 = eK \left[ w^{2q-k} + \sum_{j=1}^{\infty} \beta_j w^{2q-k-j} \right], \]

\[ 1/(f \circ g^{-1})(1/w) = w + \sum_{j=q-1}^{\infty} a_j w^{-j}, \]

where the $\beta_j$ are real and

\[ a_j = (B_{j+2} - A_{j+2}) + (B_{q+1} - A_{q+1})^2 \max(0, j - 2q + 2) \]

\[ + \sum_{r=q+1}^{j+1} (B_{r} - A_{r}) n(\nu, j) \prod_{\mu=1}^{q} B_{\mu} e(\mu; \nu, j), \]

(or a corresponding expression with $C_r = B_r$) where $q-1 \leq j \leq 2q-1$ and $n(\nu; j)$, $e(\mu; \nu, j)$ are integers. Therefore taking $m=2q$, $m-3 = 2q-3$, $k = (m-4)/2 = q-2$ and $a_{q-2} = 0$ in the General Coefficient Theorem we obtain

\[ \Re \left\{ e \left[ a_{2q-3} + \sum_{j=1}^{q-2} \beta_j a_{2q-3-j} \right] \right\} \leq 0. \]

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Consequently by (6) it follows from (7) that
\[ B_{2q-1} + \sum_{j=q+1}^{2g-2} b_j(B_j - \partial A_j) \leq \partial A_{2q-1} \]
\[ \leq C_{2q-1} + \sum_{j=q+1}^{2g-2} c_j(C_j - \partial A_j), \]
where \( b_j, c_j \) are real. By (3) we obtain
\[ B_{2q-1} \leq \partial A_{2q-1} \leq C_{2q-1}. \]
If equality occurs in (8) then it occurs in (7) and by the General Coefficient Theorem \( f \) must be at worst a translation of \( g \) along trajectories of \( \Omega \) in the \( |\Omega|^{1/2} \) -metric. But translations alter the value of \( A_2 \), and therefore \( f = g \). Since \( (\partial A_1, \ldots, \partial A_{2q-2}) \) belongs to a bounded schlicht function \( b \in S_q \) we know that \( B_{2q-1} < C_{2q-1} \). Finally the conformal isotopy \( t^{-1}f(tz) \) \((0 < t \leq 1, f \in S)\) which deforms \( f \) into the identity in \( S \) may be used to establish the remaining topological facts about \( V_{q,2q-1} \).

To prove the last statement in Theorem 1 for \( V_{q,2q-1} \) we use the existence of a continuous function \( \theta_e : V_{q,2q-1} \to H_{2q,e} \) with the following property. If \( \psi_t : H_{2q,e} \to V_{q,2q-1} \) \((e = \pm 1)\) is defined by setting \( \psi_1(\Omega*, g*) = (B_1, \ldots, B_{2q-1}), \psi_{-1}(\Omega**, f**) = (C_1, \ldots, C_{2q-1}) \), then the restriction of \( \psi_e \circ \theta_e \) to \( \partial V_{q,2q-1} \) is homotopic to the identity. Consequently it follows from Brouwer's Fixed Point Theorem that \( \psi_e(\theta_e(V_{q,2q-1})) = V_{q,2q-1} \) and hence \( \psi_e(H_{2q+1,e}) = V_{q,2q-1} \) for \( e = \pm 1 \).

The induction step is completed by merely repeating this argument to obtain the facts in Theorem 1 for \( V_{q,2q} \). Q.E.D.

A map which is an extension of \( \theta_e \) to mixed coefficient regions is defined by a straightforward but rather lengthy process in [4]. The main feature of our construction is the use of the space of induced positive quadratic differentials \( \Omega \circ f \) on \( E \) and the canonical decompositions of \( \partial E \) into pairs of identified arcs introduced by Schaeffer and Spencer [6, Chapter VIII]

**Theorem 2.** If \( f \in S_p \) then
\[ \partial A_n \leq n \quad \text{for} \quad n \leq 2p + 1 \]
with equality occurring only for one of the Koebe slit functions \( z/(1 \pm z)^2 \).

**Proof.** If \( n \leq 2p \) then Theorem 2 follows immediately from Theorem 1 (see (8)) because of the truth of the Bieberbach conjecture for functions in \( S_\infty \). If \( f \in S_{2p+1} \) then by Theorem 1 there is an \( (\Omega**, g**) \in H_{2p+1,-1} \) such that
Applying the General Coefficient Theorem as in the proof of Theorem 1 to $\Omega**, g**_1(\zeta)$ with $m = 2p + 2$, $m - 3 = 2p - 1$, $k = (m - 4)/2 = p - 1$ we obtain

\[ \Re \left\{ \left[ a_{2p-1} + \sum_{j=1}^{p} \beta_j a_{2p-1-j} + \frac{(p - 1)}{2} a_{p-1}^2 \right] \right\} \leq 0 \]

where the $a_j$ are defined in (5), (6). Using (9) to simplify (10) we obtain

\[ \Re A_{2p+1} + (\Re A_{p+1})^2(p + 1)/2 \leq C_{2p+1} \]

which implies our result.

We note that (8) shows $\Re A_n \geq -n$ if $f \in S_p$ and $n \leq 2p$. Also, note that to prove $\Re A_6 \leq 6$ we need both $A_2, A_3$ real whereas Ozawa needs only $A_2 \geq 0$.

We wish to mention that this proof was inspired by a remarkably short and elegant proof of Löwner's inequality $|A_3| \leq 3$ which was given by Jenkins [2]. One merely applies the General Coefficient Theorem [1] to the Schiffer quadratic differentials [7, p. 442, (20)] to show that the extremal functions for this problem are in $S_\infty$.

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