TERM-BY-TERM DIFFERENTIABILITY OF MERCER'S EXPANSION

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Let \( K(x, y), 0 \leq x, y \leq 1 \), be a real, symmetric, continuous and non-negative-definite kernel on \([0, 1] \times [0, 1]\). Thus, the integral operator generated by \( K \) has nonnegative eigenvalues and the orthonormalized eigenfunctions \( \lambda_i \) and \( \phi_i, i = 0, 1, 2, \cdots \). Then, according to Mercer's theorem \([1]\),

\[
K(x, y) = \sum_i \lambda_i \phi_i(x) \phi_i(y)
\]

uniformly on \([0, 1] \times [0, 1]\). This paper concerns with term-by-term differentiability of the above series while retaining the same sense of convergence. In particular, we obtain a condition, explicitly on \( K \), for such differentiability.

**Theorem.** If \( \left( \frac{\partial^{2n}}{\partial x^n \partial y^n} \right) K(x, y) \) exists and is continuous on \([0, 1] \times [0, 1]\), then \( \phi^{(n)}_i \), the \( n \)th derivative of \( \phi_i \), exists and is continuous on \([0, 1]\) for each \( i = 0, 1, 2, \cdots \), and

\[
\frac{\partial^{2n}}{\partial x^n \partial y^n} K(x, y) = \sum \lambda_i \phi^{(n)}_i(x) \phi^{(n)}_i(y)
\]

uniformly on \([0, 1] \times [0, 1]\). Conversely, if \( \phi^{(n)}_i \) exists and is continuous on \([0, 1]\), and if the series of (2) converges uniformly on \([0, 1] \times [0, 1]\), then \( \left( \frac{\partial^{2n}}{\partial x^n \partial y^n} \right) K(x, y) \) exists, is continuous and is equal to the limit of the series.

**Proof.** The method of induction will be used.

(a) **Proof of the first assertion.** First, since \( \left( \frac{\partial^{2n}}{\partial x^n \partial y^n} \right) K(x, y) \) exists and is continuous in \((x, y)\), existence and continuity of \( \phi^{(n)}_i \) can be readily established by differentiating \( n \) times both sides of

\[
\phi_i(x) = \frac{1}{\lambda_i} \int_0^1 K(x, y) \phi_i(y) dy, \quad i = 0, 1, 2, \cdots .
\]

For notational simplicity, define for \( k = 1, 2, \cdots, n \),

\[
K_k(x, y) = \frac{\partial^{2k}}{\partial x^k \partial y^k} K(x, y),
\]

\[
K^{(p)}_k(x, y) = K_k(x, y) - \sum_{i=0}^j \lambda_i \phi^{(k)}_i(x) \phi^{(k)}_i(y).
\]

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The following steps will be taken to establish the assertion for \( n = 1 \).

1°. \( R_1^{(j)}(x, x) \geq 0, \quad 0 \leq x \leq 1, \) for every \( j \).

Suppose \( R_1^{(j)}(x_0, x_0) < 0 \) for some \( x_0 \in [0, 1] \). Then it follows from continuity of \( R_1^{(j)} \) that there exists a neighborhood \( x_0 - \delta < x, y < x_0 + \delta \) where \( R_1^{(j)}(x, y) < 0 \). Thus, from (1),

\[
0 > \int_{x_0-\delta}^{x_0+\delta} \int_{x_0-\delta}^{x_0+\delta} R_1^{(j)}(x, y) \, dx \, dy = \sum_{i=j+1}^{\infty} \lambda_i \int_{x_0-\delta}^{x_0+\delta} \phi_i'(x) \, dx \int_{x_0-\delta}^{x_0+\delta} \phi_i'(y) \, dy \geq 0,
\]

a contradiction.

2°. The series of (2) with \( n = 1 \) converges uniformly in \( x \) for every fixed \( y \) and also in \( y \) for every fixed \( x \); thus its limit, denoted by \( K_1^*(x, y) \), is continuous in \( x \) for every fixed \( y \) and also in \( y \) for every fixed \( x \).

Note \( \sum \lambda_i |\phi_i'(x)|^2 \) converges since its partial sums form a non-decreasing sequence bounded by \( K_1(x, x) \) as seen from 1°. Define

\[
M = \max_{0 \leq x \leq 1} K_1(x, x),
\]

which exists since \( K_1 \) is continuous by hypothesis. Then, from Cauchy's inequality,

\[
\left| \sum_{i=m}^{n} \lambda_i \phi_i(x) \phi_i'(y) \right|^2 \leq \sum_{i=m}^{n} \lambda_i \left| \phi_i(x) \right|^2 \sum_{i=m}^{n} \lambda_i \left| \phi_i'(y) \right|^2
\]

(4)

\[
\leq M \sum_{i=m}^{n} \lambda_i \left| \phi_i(y) \right|^2.
\]

Hence, \( \sum \lambda_i \phi_i(x) \phi_i'(y) \) converges uniformly in \( x \) for every fixed \( y \). Similarly, it converges uniformly in \( y \) for every fixed \( x \).

3°. \( K_1(x, y) = K_1^*(x, y) \).

Note \( K_1 = K_1^* \), a.e. \([dxdy]\), since both \( K_1 \) and \( K_1^* \) are measurable and, from 2° and (1),

\[
\int_0^x \int_0^y [K_1(u, v) - K_1^*(u, v)] \, du \, dv = \int_0^x \int_0^y K_1(u, v) \, du \, dv - \sum_i \lambda_i \int_0^x \phi_i'(u) \, du \int_0^y \phi_i'(v) \, dv
\]

\[
= \int_0^x \int_0^y K_1(u, v) \, du \, dv - K(x, y) + K(x, 0) + K(0, y) - K(0, 0)
\]

\[
= 0
\]
for every $x$ and $y$. Then, from Fubini's theorem [2], for almost every $x$, $K_1(x, y) = K_1^*(x, y)$ for almost every $y$. But, since for every fixed $x$ both $K_1$ and $K_1^*$ are continuous in $y$, for almost every $x$ the equality holds for every $y$. Hence, for every $y$ the equality holds for almost every $x$. However, for every fixed $y$ $K_1$ and $K_1^*$ are continuous in $x$ also. Thus, the equality holds for every $x$ and $y$.

4°. The series of (2) with $n=1$ converges uniformly in $x$ and $y$ simultaneously.

From 3°,

$$K_1(x, x) = \sum \lambda_i \left| \phi_i'(x) \right|^2.$$ 

Observe that the partial sums of the series form a nondecreasing sequence of continuous functions converging to a continuous function. Hence, according to Dini's theorem, the convergence is uniform. Then, by applying Cauchy's inequality (4) again, we conclude that $\sum \lambda_i \phi_i'(x)\phi_j'(y)$ converges uniformly in $x$ and $y$ simultaneously.

Next, note in the preceding proof for $n=1$ that we have used only the continuity of $\phi_i$ and uniform convergence of (1) together with $\lambda_i \geq 0, i = 0, 1, 2, \cdots$, but not the orthonormality of $\{\phi_i\}$. Hence, upon replacement of $\phi_i$, $K$, $\phi_i'$, $K_1$, $K_1^*$ and $R_i^{(j)}$ by $\phi_i^{(k)}$, $K_k$, $\phi_i^{(k+1)}$, $K_{k+1}$, $K_{k+1}^*$ and $R_i^{(j)}$ respectively, the preceding proof establishes the assertion for $n=k+1$ if it holds for $n=k$. Therefore, by induction, the assertion holds for every $n$.

(b) Proof of the converse statement. To prove for $n=1$, note that $K_1^*(x, y)$ is continuous in both $x$ and $y$ since, by hypothesis, the series of (2) with $n=1$ converges uniformly in $x$ and $y$ simultaneously. Note also that

$$\int_0^x \int_0^y K_1^*(u, v) \, du \, dv = \sum \lambda_i \int_0^x \phi_i'(u) \, du \int_0^y \phi_j'(v) \, dv$$

$$= K(x, y) - K(x, 0) - K(0, y) + K(0, 0),$$

where the second equality follows from (1). Now, from (3), differentiability of $\phi_i$ implies that of $K(x, 0)$ and $K(0, y)$. Thus, differentiability of the left-hand side of (5) with respect to $y$ and then $x$, implies existence of $\partial^2/(\partial x \partial y))K(x, y)$. Hence, upon differentiation of both sides of (5),

$$K_1^*(x, y) = \frac{\partial^2}{\partial x \partial y} K(x, y).$$

Through a similar argument, we establish the converse statement.
for \( n = k + 1 \) if it holds for \( n = k \). Hence, by induction, it holds for every \( n \).

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**References**


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**SOME GENERALIZATIONS OF OPIAL'S INEQUALITY**

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The inequality \( \int_0^a uu' \leq a/2\int_0^a |u'|^2 \) which is valid for absolutely continuous \( u \) with \( u(0) = 0 \) has received successively simpler proofs by Opial, [5], Olech [4], Beesack [1], Levinson [2], Pederson [6], and Mallows [3]. It is the purpose of this paper to use the method of Olech to obtain some more general inequalities.

**Theorem 1.** Let \( u \) be absolutely continuous on \((a, b)\) with \( u(a) = 0 \), where \(-\infty \leq a < b < \infty\). Let \( f(t) \) be a continuous, complex function defined for all \( t \) in the range of \( u \) and for all real \( t \) of the form \( t(s) = \int_0^s |u'(x)| \, dx \). Suppose that \( |f(t)| \leq f(|t|) \), for all \( t \), and that \( f(t_1) \leq f(t_2) \) for \( 0 \leq t_1 \leq t_2 \). Let \( r \) be positive, continuous and in \( L^{1/q}[a, b] \), where \( 1/p + 1/q = 1 \), \( p > 1 \). Let \( F(s) = \int_0^s f(x) \, dx \), \( s > 0 \). Then

\[
\int_a^b |f(u)u'| \, dx \leq F \left[ \left( \int_a^b r^{1-q} \right)^{1/q} \left( \int_a^b r \, u'|^p \right)^{1/p} \right]
\]

with equality iff \( u(x) = A\int_0^x r^{1-q} \). The same result (but with equality for \( u(x) = \int_0^x r^{1-q} \)) holds if \( u(b) = 0 \) and \(-\infty < a < b \leq \infty\).

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